A geometric approach for constructing SRB measures in hyperbolic dynamics

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R. Bowen

Methods of Symbolic Dynamics

Collection of papers

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"Advances in Foreign Science Series", Mathematics
Editors: A. Kolmogorov and S. Novikov

1. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Mathematics, **470**, Springer-Verlag, 1975
2. R. Bowen, *Markov Partitions and Minimal Sets for Axiom A Diffeomorphisms*, American Journal of Mathematics, **92** (1970) 907–917
3. R. Bowen, *Symbolic Dynamics for Hyperbolic Flows*, American Journal of Mathematics, **95** (1973) 429–459
4. R. Bowen and D. Ruelle *The Ergodic Theory of Axion A Flows*, Inventiones Mathematicae, **29** (1975) 181–202
5. R. Bowen, *A Horseshoe of Positive Measure*, Inventiones Mathematicae, **29** (1975) 203–204
6. R. Bowen, *Topological Entropy for Noncompact Sets*, Transactions of the AMS, **184** (1973) 125–136
7. Appendix by V. Alekseev and M. Jakobson, *Symbolic Dynamics and Hyperbolic Dynamical Systems*, 196–240
8. Full List of Bowen's papers, 241

“This book had already been sent for production, when we learned that its author [R. Bowen] had passed away – the news whose tragedy was aggravated by its complete unexpectedness. In the prime of his creativity and physical strength died one of the most active and original mathematicians of the last decade [the 70th].

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I was not acquainted with him personally – only by his work which I value highly. Accounts by our American colleagues describe him not only as a scientist but as a joyful and sociable man."

V. M. Alexseev

Topological Attractors

1. $f: M \to M$ a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M;
2. $U \subset M$ an open subset with the property that $\overline{f(U)} \subset U$; such a set U is called a trapping region;
3. $\Lambda = \bigcap_{n\geq 0} f^n(U)$ a topological attractor for f; we allow the case $\Lambda = M$.

Λ is compact, f-invariant, and locally maximal (i.e., if $\Lambda' \subset U$ is invariant, then $\Lambda' \subset \Lambda$).

Physical Measures

μ an invariant probability measure on Λ. The basin of attraction of μ is

$$B_\mu = \left\{ x \in U : \frac{1}{n}\sum_{k=0}^{n-1} h(f^k(x)) \to \int_\Lambda h\, d\mu \text{ for any } h \in C^1(M) \right\}$$

μ is a physical measure if $m(B_\mu) > 0$. An attractor with a physical measure is often referred to as a Milnor attractor.

Hyperbolic Measures

Let μ be a hyperbolic measure on an attractor Λ, i.e., a measure with nonzero Lyapunov exponents (with some being positive and some being negative). Results of nonuniform hyperbolicity theory allow one to construct for almost every $x \in \Lambda$ stable $E^s(x)$ and unstable $E^u(x)$ subspaces which integrate locally into local stable $V^s(x)$ and local unstable $V^u(x)$ manifolds. The latter give rise to global unstable $W^u(x)$ manifolds

$$W^u(x) := \bigcup_{n\in\mathbb{Z}} f^n(V^u(f^{-n}(x))).$$

It is easy to see that for such points x we have $W^u(x) \subset \Lambda$, so that the attractor contains all the global unstable manifolds of its points. On the other hand the intersection of Λ with local stable manifolds of its points may be a Cantor set.

A hyperbolic measure μ on Λ is an SRB measure if for almost every $x \in \Lambda$, the conditional measure $\mu^u(x)$ generated by μ on the unstable leaf $V^u(x)$ is absolutely continuous with respect to the leaf-volume $m^u(x)$ on $V^u(x)$.

SRB Measures

A hyperbolic measure μ on Λ is an SRB measure if for almost every $x \in \Lambda$, the conditional measure $\mu^u(x)$ generated by μ on the unstable leaf $V^u(x)$ is absolutely continuous with respect to the leaf-volume $m^u(x)$ on $V^u(x)$.

Using results of nonuniform hyperbolicity theory, one can obtain a sufficiently complete description of ergodic properties of an SRB measure μ (Ledrappier):

- Ergodic components of μ are of positive measure.
- On each ergodic component μ is Bernoulli up to a rotation.
- Every SRB measure μ is a physical measure, i.e., $m(B_\mu) > 0$ (but not every physical measure is an SRB measure).
- There may exist at most countably many ergodic SRB measures.

Hyperbolic Attractors

An attractor Λ is (uniformly) hyperbolic if $f|\Lambda$ is a uniformly hyperbolic set (which is of course, locally maximal).

Theorem (Sinai, Ruelle, Bowen)

Assume that f is $C^{1+\alpha}$ and that Λ is a uniformly hyperbolic attractor. The following statements hold:

1. *There is an SRB measure μ for f on Λ.*
2. *There are at most finitely many ergodic SRB measures for f on Λ.*
3. *If $f|\Lambda$ is topologically transitive, then there is a unique SRB-measure μ for f on Λ.*

1. Sinai (*Markov Partitions and Y-diffeomorphisms*, 1968) proved this theorem in the case $\Lambda = M$, i.e., when f is an Anosov diffeomorphism. His proof uses Markov partitions. If $\mathcal{R}$ is such a partition of small diameter, then consider the partition $\mathcal{R}^- = \bigvee_{n=0}^{\infty} f^{-n}\mathcal{R}$ which satisfies:

- $f\mathcal{R}^- \geq \mathcal{R}^-$;
- $\bigwedge_{k=0}^{\infty} f^k\mathcal{R}^-$ is the trivial partition;
- there is $r > 0$ such that every element of the partition $\mathcal{R}^-$ contains a ball of radius r.

Note that f^{-n} transfers the normalized leaf-volume on $C_{f^n(\mathcal{R}^-)}(f^n(x))$ to a measure μ_n on $C_{\mathcal{R}^-}(x)$ which is equivalent to the leaf-volume on $C_{\mathcal{R}^-}(x)$ with density $\rho_n(y)$. The proof then goes by showing that the sequence of functions ρ_n converges uniformly to a continuous function $\tilde{\rho}(y) = \tilde{\rho}_{C_{\mathcal{R}^-}(x)}(y)$, which can be viewed as the density function for a normalized measure $\tilde{\mu}_{C_{\mathcal{R}^-}(x)}$ on $C_{\mathcal{R}^-}(x)$. These measures satisfy the relation:

$$\tilde{\mu}(A|C_{f^{-1}(\mathcal{R}^-)}) = \tilde{\mu}(A|C'_{\mathcal{R}^-})\tilde{\mu}(C'_{\mathcal{R}^-}|C_{f^{-1}(\mathcal{R}^-)})$$

which can be used to show that for any $x \in M$ and any measurable subset $A \subset M$ the limit

$$\mu(A) = \lim_{n\to\infty} \tilde{\mu}(A|C_{f^{-n}(\mathcal{R}^-)}(x)$$

exists and does not depend on x. The number $\mu(A)$ determines an invariant measure for f which is the desired SRB measure.

2. Ruelle's approach (*A Measure Associated with Axiom-A Attractors*, 1976) produces the desired SRB measure μ as an equilibrium measure for the geometric potential $\varphi(x) := -\log(\mathrm{Jac}(df|E^u(x))$ that is a measure $\mu = \mu_\varphi$ for which

$$0 = h_{\mu_\varphi} + \int_\Lambda \varphi\, d\mu_\varphi = \sup(h_\mu + \int_\Lambda \varphi\, d\mu),$$

where the supremum is taken over all invariant measures on Λ. The proof of this fact uses Markov partitions and the corresponding symbolic representation of the map by a subshift of finite type. This approach is described with great details in Bowen's paper *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (1975).

3. Bowen's proof of the theorem (*Some systems with unique equilibrium states*, 1974) uses the specification property of f which roughly speaking means that for every finite number of orbit segments one can find a single periodic orbit, which consecutively approximates each segment with a fixed precision and (uniformly) bounded transition times.

The proof goes by considering a weak* limit point μ of the following sequence of f-invariant Borel probability measures

$$\mu_n := \frac{1}{Z_n(\varphi)} \sum_{x \in \mathrm{Per}_n} \exp(S_n\varphi(x))\delta_x,$$

where δ_x is the Dirac measure at x, Per_n is the set of periodic points of period n, and

$$Z_n(\varphi) := \sum_{x \in \mathrm{Per}_n} \exp(S_n\varphi(x)), \quad S_n\varphi(x) := \sum_{k=0}^{n-1} \varphi(f^k(x)).$$

The specification property guarantees that every such μ has the Gibbs property from which one concludes that μ is the unique equilibrium measure for φ. Applying this result to the geometric potential yields the desired SRB measure.

After the classical work of Sinai, Ruelle and Bowen there has been a great body of activity on constructing SRB measures for systems whose level of hyperbolicity is weaker than uniform (the list is incomplete and does not include work on acim in one-dimensional dynamics):

1. Young diffeomorphisms: Young, *Statistical properties of dynamical systems with some hyperbolicity (1998)* – a class of maps that allow symbolic representation by a tower of a special type – Young tower; examples include Hénon attractors (Benedics-Young) and some more general families of maps with one unstable direction (Young-Wang).

2. Partially hyperbolic maps and maps with dominated splitting:
   - Alves, Bonatti, and Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding* (2000);
   - Bonatti, Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting* (2000);
   - Alves, Dias, Luzzatto, and Pinheiro, *SRB measures for partially hyperbolic systems whose central direction is weakly expanding* (2016);
   - Burns, Dolgopyat, Pesin, and Pollicott, *Stable ergodicity for partially hyperbolic attractors with negative central exponents* (2008);
   - Vásquez, *Stable ergodicity for partially hyperbolic attractors with positive central Lyapunov exponents* (2009).

3. Countable Markov partitions, Sarig, *Symbolic dynamics for surface diffeomorphisms with positive entropy* (2013) and Ben Ovadia, *Symbolic dynamics for non-uniformly hyperbolic diffeomorphisms of compact smooth manifolds*, Preprint, 2017.
4. Non-uniform specification, Climenhaga-Thomson, *Intrinsic ergodicity beyond specification: β-shifts, S-gap shifts, and their factors* (2012) and *Unique equilibrium states for flows and homeomorphisms with non-uniform structure* (2016); This approach is used to study equilibrium measures for geodesic flows: Burns, Climenhaga, Fisher, and Thompson, *Unique equilibrium states for geodesic flows in nonpositive curvature* (2017).
5. Jakobson, *Thermodynamic formalism for some systems with countable Markov structures* (2017).

An overview of the geometric approach

The idea of the geometric approach is to follow the classical Bogolyubov-Krylov procedure for constructing invariant measures by pushing forward a given reference measure. In our case the natural choice of a reference measure is the Riemannian volume m restricted to the neighborhood U, which we denote by m_U. We then consider the sequence of probability measures

$$\mu_n := \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m_U.$$

Any weak* limit point of this sequence of measures is called a natural measure and while in general, it may be a trivial measure, under some additional hyperbolicity requirements on the attractor one obtains an SRB measure. For hyperbolic attractor this was proved by Sinai and Ruelle.

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To this end consider a point $x \in \Lambda$, its local unstable manifold $V^u(x)$, and the leaf-volume $m^u(x)$. It can be extended to a measure $\tilde{m}^u(x)$ on Λ by setting $\tilde{m}^u(x)(E) := m^u(x)(E \cap V^u(x))$ for any Borel set $E \subset \Lambda$. If we use this measure as the reference measure, then the above sequence of measures $\mu_n = \mu_n(x)$ converges to the SRB measure on Λ.

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The idea of describing an invariant measure by its conditional probabilities on the elements of a continuous partition goes back to the classical work of Kolmogorov and later work of Dobrushin on random fields. It was also used by Sinai and P. to establish existence of u-measures – an analog of SRB measures – for partially hyperbolic attractors.

Outline of Proof of Sinai-Ruelle-Bowen Theorem

Fix $x \in \Lambda$ and let $W := V^u(x)$. It can be represented as $W = \exp_x(\text{graph}\psi)$, where $\psi : B^u(0,r) \to E^s(x)$ is $C^{1+\alpha}$ and satisfies $\psi(0) = 0$ and $d\psi(0) = 0$ (here $B^u(0,r)$ is the ball in $E^u(x)$ centered at the origin of radius r). Fix constants $\mathbf{I} = (\gamma, \kappa, r)$ and consider the space $\mathcal{R}_{\mathbf{I}}$ of local unstable manifolds of size r satisfying $\|d\psi\| \le \gamma$, $|d\psi|_\alpha \le \kappa$.

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Given $W \in \mathcal{R}_{\mathbf{I}}$, consider a standard pair (W, ρ) where ρ is a Hölder continuous density function on W. The idea of working with standard pairs was introduced by Chernov and Dolgopyat and is an important tool in dynamic and in particular, in studying SRB measures via geometric techniques.

Fix $L > 0$, write $\mathbf{K} = (\mathbf{I}, L) = (\gamma, \kappa, r, L)$, and consider the space of standard pairs

$$\mathcal{R}'_{\mathbf{K}} = \{(W,\rho)\colon W \in \mathcal{R}_{\mathbf{I}},\ \rho \in C^\alpha(W, [\frac{1}{L}, L]),\ |\rho|_\alpha \le L\}.$$

The spaces $\mathcal{R}_{\mathbf{I}}$ and $\mathcal{R}'_{\mathbf{K}}$ are compact in the natural product topology.

Each measure η on $\mathcal{R}'_{\mathbf{K}}$ determines a measure $\Phi(\eta)$ on Λ by

$$\Phi(\eta)(E) = \int_{\mathcal{R}'_{\mathbf{K}}} \int_{E\cap W} \rho(x)\, dm_W(x)\, d\eta(W,\rho).$$

Write $\mathcal{M}(\Lambda)$ and $\mathcal{M}(\mathcal{R}'_{\mathbf{K}})$ for the spaces of finite Borel measures on Λ and $\mathcal{R}'_{\mathbf{K}}$, respectively. One can show that the map $\Phi\colon \mathcal{M}(\mathcal{R}'_{\mathbf{K}}) \to \mathcal{M}(\Lambda)$ is continuous; in particular, the set $\mathcal{M}_{\mathbf{K}} = \Phi(\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{K}}))$ is compact, where we write $\mathcal{M}_{\leq 1}$ for the space of measures with total weight at most 1.
On a uniformly hyperbolic attractor, an invariant probability measure is an SRB measure if and only if it is in $\mathcal{M}_{\mathbf{K}}$ for some $\mathbf{K}$.
It remains to show that $\mathcal{M}_{\mathbf{K}}$ is invariant under f_*.

To this end consider the images $f^n(W)$ and observe that for each n, the measure $f^n_* m_W$ is absolutely continuous with respect to leaf volume on $f^n(W)$. For every n, the image $f^n(W)$ can be covered with uniformly bounded multiplicity (which depends only on the dimension W) by a finite number of local manifolds W_i, so that

$$f^n_* m_W \text{ is a convex combination of measures } \rho_i \, dm_{W_i},$$

where ρ_i are Hölder continuous positive densities on W_i. This requires a version of the Besicovitch covering lemma, which is usually formulated for geometrical balls, so one must choose the W_i in such a way that each $f^n(W_i)$ is "sufficiently close" to being a ball in $f^n(W)$.

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We conclude that with an appropriate choice of parameters $\mathbf{K}$, the space $\mathcal{M}_{\mathbf{K}}$ is invariant under the action of f_*.

Non-uniformly Hyperbolic Attractors

If μ is an SRB measure, then every point in the basin of attraction B_μ has non-zero Lyapunov exponents. A natural and interesting question is whether the converse holds true. In some sense this is stated in the following conjecture by Viana (*Dynamics: a probabilistic and geometric perspective*, ICM, Berlin (1998):

Conjecture: If a smooth map has only non-zero Lyapunov exponents at Lebesgue almost every point, then it admits an SRB measure.

The result that I will describe below is an attempt to better understand this conjecture.

Effective hyperbolicity

We make the following standing assumptions:

(H1) f is $C^{1+\alpha}$, Λ a topological attractor for f, and U a trapping region.

(H2) There exists a forward-invariant set $A \subset U$ of positive volume with two measurable cone families $K^s(x) = K(x, E^s(x), a^s(x))$ and $K^u(x) = K(x, E^u(x), a^u(x))$ in T_xM such that

- $\overline{Df(K^u(x))} \subset K^u(f(x))$ for all $x \in A$;
- $\overline{Df^{-1}(K^s(f(x)))} \subset K^s(x)$ for all $x \in f(A)$.
- $T_xM = E^s(x) \oplus E^u(x)$; moreover $d_s = \dim E^s(x)$ and $d_u = \dim E^u(x)$ do not depend on x.

Such cone families automatically exist if f is uniformly hyperbolic on Λ but in our setting $K^{s,u}$ are assumed to be only measurable and the families of subspaces $E^{u,s}(x)$ are not necessarily invariant.

Define:

$$\lambda^u(x) := \inf\{\log \|Df(v)\| \mid v \in K^u(x), \|v\| = 1\},$$
$$\lambda^s(x) := \sup\{\log \|Df(v)\| \mid v \in K^s(x), \|v\| = 1\}.$$

If the splitting $E^s \oplus E^u$ is dominated, $\lambda^s(x) < \lambda^u(x)$ for every x.

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If the splitting $E^s \oplus E^u$ is dominated, $\lambda^s(x) < \lambda^u(x)$ for every x. Define the defect from domination at x to be

$$\Delta(x) = \frac{1}{\alpha} \max(0,\ \lambda^s(x) - \lambda^u(x)),$$

where $\alpha \in (0, 1]$ is the Hölder exponent of df. Roughly speaking, $\Delta(x)$ controls how much the curvature of unstable manifolds can grow as we go from x to $f(x)$. Indeed, if $\lambda^s(x) > \lambda^u(x)$, then the action of df can push tangent vectors away from E^u and towards E^s, so that the image of an unstable (or admissible) manifold can curl up under the action of f, and $\Delta(x)$ quantifies how much this can happen.

The following quantity is positive whenever f expands vectors in $K^u(x)$ and contracts vectors in $K^s(x)$:

$$\lambda(x) = \min(\lambda^u(x) - \Delta(x), -\lambda^s(x)).$$

Denote the angle between the boundaries of $K^s(x)$ and $K^u(x)$ by

$$\theta(x) = \inf\{\angle(v, w)\colon v \in K^u(x), w \in K^s(x)\}.$$

We say that a point $x \in A$ is effectively hyperbolic if the following two conditions hold:

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k(x)) > 0, \tag{1}$$

that is not only are the Lyapunov exponents of x positive for vectors in K^u and negative for vectors in K^s, but λ^u gives enough expansion to overcome the "defect from domination" given by Δ.

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$$\lim_{\bar\theta\to 0} \limsup_{N\to\infty} \frac{1}{N} \#\{n \in [0, N] : \theta(f^n(x)) < \bar\theta\} = 0, \tag{2}$$

that is the frequency with which the angle between the stable and unstable cones drops below a specified threshold $\bar\theta$ can be made arbitrarily small by taking the threshold to be small.

If Λ is a hyperbolic attractor for f, then every point $x \in U$ is effectively hyperbolic.

Theorem (Climenhaga, Dolgopyat, P.)

Let f be a $C^{1+\alpha}$ diffeomorphism of a compact manifold M, and Λ a topological attractor for f. Assume that

1. *f admits measurable invariant cone families as in (H2);*
2. *the set $S \subset A$ of effectively hyperbolic points has positive volume, $m(S) > 0$.*

Then f has an SRB measure supported on Λ.

A similar result can be formulated given information about the set of effectively hyperbolic points on a single local unstable admissible submanifold.

Theorem (Climenhaga, Dolgopyat, P.)

Let f be a $C^{1+\alpha}$ diffeomorphism of a compact manifold M, and Λ a topological attractor for f. Assume that

1. *f admits measurable invariant cone families as in (H2);*
2. *there is a d_u-dimensional embedded submanifold $W \subset U$ such that $m_W(\{x \in S \cap W : T_xW \subset K^u(x)\}) > 0$.*

Then f has an SRB measure supported on Λ.

This theorem covers essentially all known situations where existence of SRB measures was shown (e.g., uniformly and partially hyperbolic attractors, and attractors with dominated splitting). It can also be used to study some new examples.

Let f be a $C^{1+\alpha}$ diffeomorphism with a hyperbolic attractor Λ which has one-dimensional unstable bundle. We perturb f near a hyperbolic fixed point $p \in \Lambda$ to obtain a new map g that has an indifferent fixed point at p. The case when M is two-dimensional and f is volume-preserving was studied by Katok. We allow manifolds of arbitrary dimensions and dissipative maps. For example, one can choose f to be the Smale–Williams solenoid or its sufficiently small perturbation.

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Suppose that there is a neighborhood $U \ni p$ with local coordinates in which f is the time-1 map of the flow generated by $\dot{x} = Ax$ for some matrix A. Assume that the local coordinates identify the splitting $E^u \oplus E^s$ with $\mathbb{R} \oplus \mathbb{R}^{d-1}$, so that $A = A_u \oplus A_s$, where $A_u = \gamma \mathrm{Id}_u$ and $A_s = -\beta \mathrm{Id}_s$ for some $\gamma, \beta > 0$.

Using local coordinates in U, we identify p with 0. Fix $0 < r_0 < r_1$ such that $B(0, r_1) \subset U$, and let ψ be a $C^{1+\alpha}$ function such that

1. $\psi(x) = \|x\|^\alpha$ for $\|x\| \leq r_0$;
2. $\psi(x) = 1$ for $\|x\| \geq r_1$;
3. $\psi(x) > 0$ for $x \neq 0$ and $\psi'(x) > 0$.

Let $\mathcal{X}$ be the vector field in U given by $\mathcal{X}(x) = \psi(x)Ax$. Let also g be the time-1 map of the flow in U generated by $\mathcal{X}$ and $g = f$ outside U. Note that g is $C^{1+\alpha}$.

Theorem (Climenhaga, Dolgopyat, P.)

The map g has an SRB measure.

Since g has a neutral fixed point at P, it is not uniformly hyperbolic nor it admits a dominated splitting.

Outline of Proof

The proof follows the same ideas as in the case of uniformly hyperbolic attractors, but there are some major obstacles to overcome:

- There are *a priori* no local unstable manifolds in Λ.

This can be dealt with by considering local unstable admissible manifolds $W(x)$ of a given size r along effectively hyperbolic trajectories in a neighborhood of Λ; such a manifold is given as $W(x) = \exp_x(\text{graph}\psi)$ where ψ is as above.

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- The action of f along admissible manifolds is not necessarily uniformly expanding.

This can be dealt with by considering only local unstable admissible manifolds whose geometry (i.e., the size and curvature) and dynamics (i.e., the contraction rates when moving backward) are controlled by a collection of appropriately chosen parameters – good admissible manifolds.

- Given a local unstable admissible manifold W and $n > 0$ it is no longer necessarily the case that $f^n(W)$ contains any admissible manifold of size r, let alone that it can be covered by them. When $f^n(W)$ contains some admissible manifolds of size r, we will need to control how much of it can be covered.

To deal with this obstacle we observe that given an effectively hyperbolic trajectory $\{f^n(x)\}$ and a good admissible manifold – there is a sequence of effectively hyperbolic times n_k of positive (lower) density such that for each k the manifold $f^{n_k}(W)$ contains a good admissible manifold $\tilde{W}$ around $f^{n_k}(x)$. This statement is a non-stationary version of the classical Hadamard-Perron theorem (Climenhaga, P.). Note that effectively hyperbolic times form a subset of hyperbolic times (introduced by Alves), which provide an important tool in studying genericity of Lyapunov exponents in the C^1 topology (Mané, Bochi, Viana).

We obtain that starting from a good admissible manifold W for all sufficiently large n a large part of the set $\bigcup_{j=0}^{n} f^j(W)$ can be covered by good admissible manifolds. This implies that the corresponding set of standard pairs carries a uniformly large part of the measure μ_n that is $\mu_n = \mu_n^{(1)} + \mu_n^{(1)}$ where $\|\mu_n^{(1)}\| \geq \varepsilon$ and $\mu_n^{(1)} \leq \mu_n$. Moreover, the conditional measures generated by $\mu_n^{(1)}$ on good admissible manifolds are absolutely continuous with respect to the leaf-volume. For some subsequence ℓ_j, we have that $\mu_{\ell_j} \to \mu$ and $\mu_{\ell_j}^{(1)} \to \mu^{(1)}$ where $\mu^{(1)}$ is a finite non-trivial measure on Λ, $\mu^{(1)} \leq \mu$ and $\mu^{(1)}$ is a non-invariant SRB measure. With some more work one can show that μ has an ergodic component which is an invariant SRB measure.