Unique equilibrium states for geodesic flows in nonpositive curvature

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Joint work with Keith Burns, Todd Fisher, and Daniel J. Thompson

Overview of talk

Goal: study uniqueness of equilibrium states for geodesic flows

Known results:

Curvature < 0: unif hyperbolic, unique eq state $\forall$ Hölder φ

≤ 0: non-unif hyp, unique MME

New results: (Burns–C.–Fisher–Thompson, arXiv:1703.10878)

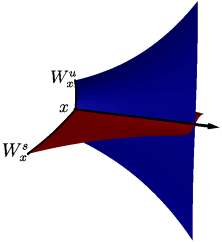
Curvature ≤ 0:

1. Unique equilibrium state if $P(\mathrm{Sing}, \varphi) < P(\varphi)$
2. Pressure gap condition is optimal and common

Motivation: Anosov systems have many invariant measures

Let M be a compact manifold, $f_t \colon M \to M$ a $C^{1+\alpha}$ Anosov flow

- $\exists$ inv. splitting $T_x M = E^u_x \oplus E^s_x \oplus E^0_x$ with $\frac{d}{dt} f_t(x) \in E^0_x$ and $C, \lambda > 0$ such that $\|Df_t|_{E^s_x}\|, \|Df_{-t}|_{E^u_x}\| \leq Ce^{-\lambda t}$ for all $t \geq 0$
- The distributions $E^{u,s,0}_x$ are Hölder continuous and integrate to foliations $W^{u,s,0}$ with local product structure



> *Study statistical behaviour:* Fix an invariant measure and study ergodic theory of measure-preserving flow (M, f_t, μ)

f_t Anosov $\Rightarrow \mathcal{M}_f = \{$flow-inv. Borel probability measures on $M\}$ is enormous, so we must identify 'distinguished' measures

- Measure of maximal entropy (MME) – maximum complexity
- Sinai–Ruelle–Bowen (SRB) measure – physically relevant

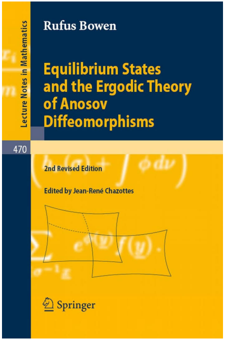
Goal: study uniqueness of equilibrium states

Equilibrium state (ES) for $\varphi\colon M \to \mathbb{R}$
achieves $\sup_\mu \left(h_\mu(f_1) + \int \varphi \, d\mu \right) =: P(\varphi)$

- $\varphi(x) = 0 \rightsquigarrow$ MME
- $\varphi^{\mathrm{geo}}(x) = -\log |\det Df|_{E^u_x}| \rightsquigarrow$ SRB

Existence? Uniqueness? Ergodic properties?

- Existence free if $\mu \mapsto h_\mu(f)$ upper semicts
- We focus on uniqueness



Theorem (Sinai, Ruelle, Bowen 1970s)

Topologically mixing Anosov system $\Rightarrow$ every Hölder $\varphi\colon M \to \mathbb{R}$ has a unique ES μ_φ. For diffeos, μ_φ is Bernoulli, has EDC + CLT.

Statistical properties for flows a little more subtle. . .

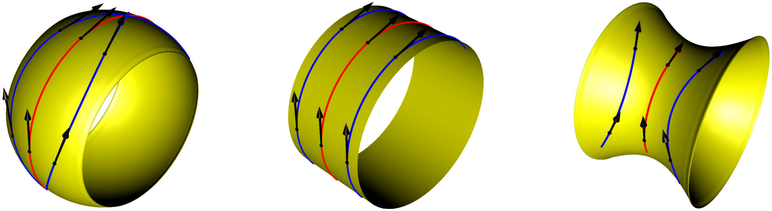
Example: Geodesic flow, dynamics controlled by curvature

Let M be a smooth compact Riemannian manifold

- $v \in T^1M \rightsquigarrow$ unique unit speed geodesic $\gamma_v(t)$ with $\dot\gamma_v(0) = v$
- Geodesic flow $f_t\colon T^1M \to T^1M$ takes $v \mapsto \dot\gamma_v(t)$

Preserves smooth Liouville measure: $(M\text{-vol}) \times (S^{d-1}\text{-vol})$

dim 2: *Given* $v \approx w$, *let* $\rho(t) =$ *distance between* $\gamma_v(t)$, $\gamma_w(t)$, *and* $\kappa(t) =$ *Gaussian curvature at* $\gamma_v(t)$; *then* $\ddot\rho \approx -\kappa\rho$ (Jacobi fields)

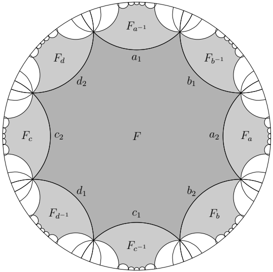


Positive curvature **concave** · Zero curvature **linear** · Negative curvature **convex**

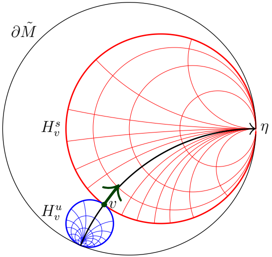
Negative curvature: hyperbolicity via $\partial\tilde{M}$, horospheres

If M has negative curvature, then the geodesic flow $f_t\colon T^1M \to T^1M$ is topologically mixing and Anosov. Every Hölder potential has a unique equilibrium state (+ Bernoulli, EDC, CLT).

1. Go to universal cover $\tilde{M}$



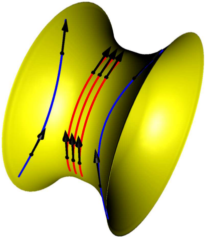
2. Get $E^{s,u}$, $W^{s,u}$ from horospheres



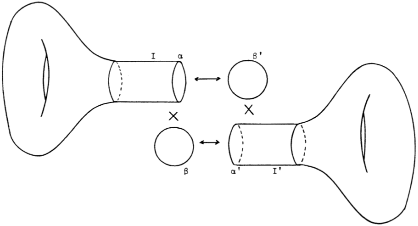
$\partial\tilde{M}$ =ideal boundary, then {geodesics on $\tilde{M}$} $\leftrightarrow$ $(\partial\tilde{M})^2\setminus$diagonal

Nonpositive curvature: two important examples

Now suppose M has nonpositive curvature; some sectional curvatures may vanish, but can never be positive.



Example 1: take surface of negative curvature, flatten near a periodic orbit



[Picture: Ballmann, Brin, Eberlein]

Dim > 2: Other possibilities

Gromov's example: 3-dim

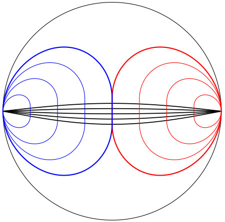
Some sectional curvature $= 0$ at every point

No neg. curved metric

Partition into singular (non-hyp) and regular (hyp) parts

Still have universal cover, horospheres, $E^{s,u}$, ...
but now M can have singular geodesics with the following (equivalent) properties:

1. $\exists$ non-trivial parallel Jacobi field
2. Horospheres have higher-order tangency
3. $E^{s,u}$ no longer transverse



$$\text{Sing} = \{v \in T^1 M : \gamma_v \text{ is singular}\} \qquad \text{Reg} = T^1 M \setminus \text{Sing}$$

$$\mu \in \mathcal{M}_f \text{ is hyperbolic (all Lyapunov exp. } \neq 0) \text{ iff } \mu(\text{Reg}) = 1$$

M is rank 1 if Reg $\neq \emptyset$; then Reg is open, dense, and invariant

- Example 1: Sing is a union of (possibly degenerate) flat strips
- Gromov's example: central strip + all orbits staying in one half

Unique MME and entropy gap in nonpositive curvature

Geodesic flow in nonpositive curvature is entropy-expansive, so every continuous φ has at least one ES. What about uniqueness?

Theorem (Knieper 1998)

If M has rank 1, then it has a unique MME μ. The MME μ is fully supported and is the limiting distribution of periodic orbits.

Guarantees entropy gap $h_{\mathrm{top}}(\mathrm{Sing}) < h_{\mathrm{top}}(T^1 M)$.

- Automatic in dim 2. In higher dimensions gap can be small; modify Gromov's example to have arbitrarily long 'neck'

Theorem (Babillot 2002; Ledrappier, Lima, Sarig 2016)

The Knieper measure is mixing; if $\dim M = 2$ then it is Bernoulli.

Open question: What about decay of correlations?

New results: unique equilibrium states and pressure gap

Theorem (Burns, C., Fisher, Thompson 2017)

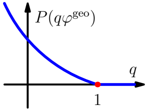
Let M be rank 1, and $\varphi\colon T^1M \to \mathbb{R}$ be Hölder or $q\varphi^{\mathrm{geo}}$ $(q \in \mathbb{R})$.

1. *If $P(\mathrm{Sing}, \varphi) < P(\varphi)$, then φ has a unique eq. state; it is fully supported and the limit distribution of φ-weighted per. orbits.*
2. *The pressure gap holds for the following classes of potentials.*
   - *Any dim: φ is (almost) locally constant on nbhd of* Sing
     $\rightsquigarrow$ dim $M = 2$, *analytic metric: generic φ (C^0-open, C^0-dense)*
   - dim $M = 2$ *and $\varphi = q\varphi^{\mathrm{geo}}$ for any $q \in (-\infty, 1)$* $(\star)$

Moreover, the unique ES in the theorem is mixing (in preparation)

Gap necessary: if $P(\mathrm{Sing}, \varphi) = P(\varphi)$, $\exists$ singular ES

$(-\infty, 1)$ optimal in $(\star)$: $\exists$ singular ES $\forall q \geq 1$



Approach I: Markov partitions, Banach spaces, eigendata

– Get ES via eigendata of linear operator –

Anosov diffeos $\rightsquigarrow$ subshifts of finite type via Markov partitions (Sinai 1968, Bowen 1972)

Unique MME: Parry measure via eigendata of transition matrix

SFT + Hölder $\varphi \rightsquigarrow$ quasi-compact transfer operator on $C^\alpha(\Sigma^+)$ (Ruelle's Perron–Frobenius theorem, 1968)

Unique ES described by eigendata of transfer operator

Anosov flow $\rightsquigarrow$ suspension flow over SFT

- Gets unique ES for Hölder φ + strongest statistical properties
- Exponential decay of correlations for geodesic flows in negative curvature: build Banach space directly (Liverani 2004)

Approach II: Geometric, conditional measures on $W_x^{s,u}$, $\partial\tilde{M}$

– Get ES via conditional measures with appropriate scaling –

Anosov flows: Margulis measure (1970) is the unique MME

1. Build measures μ_x^u on W_x^u such that $\mu_{f_t x}^u = e^{h_{\text{top}}(f_t)}(Df_t)_*\mu_x^u$
2. Similarly on W_x^s, then take (local) product of μ_x^u, μ_x^s, Leb

μ is K (uses product structure), controls growth of periodic orbits

Negative curvature: {geodesics on $\tilde{M}$} $\leftrightarrow (\partial\tilde{M})^2\setminus$diagonal

- $\exists$ Patterson–Sullivan $\nu \in \mathcal{M}(\partial\tilde{M})$ s.t. MME $\leftrightarrow \nu \times \nu$
- $W_x^{u,s} \leftrightarrow$ horospheres $\leftrightarrow \partial\tilde{M}$ gives $\mu_x^{u,s} \leftrightarrow \nu$

See also Hamenstädt, Hasselblatt, Kaimanovich 1989/90

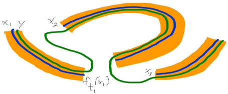
Eq. states with $\varphi \neq 0$: see Paulin, Pollicott, Schapira (2015)

C.–Pesin–Zelerowicz (in progress): build conditional measures $\mu_{\varphi,x}^u$ using Pesin–Pitskel' generalization of Bowen's 'noncpt entropy'

Approach III: Specification property

– Get ES with bare hands via proof of variational principle –

$$P(\varphi) = \lim_{\varepsilon\to 0} \overline{\lim_{T\to\infty}} \frac{1}{T} \log \sup_{\substack{E\subset X \\ (T,\varepsilon)\text{-sep}}} \sum_{x\in E} e^{\int_0^T \varphi(f_t x)\,dt}$$



Theorem (Bowen 1972, 1974)

If $\{f_t\}$ is an Anosov flow and μ_T is equidistributed on periodic orbits of length $\leq T$, then $\mu_T \to$ unique MME as $T \to \infty$.

Uses **specification property**: $\forall$ shadowing scale $\varepsilon > 0$ $\exists$ gap size $\tau > 0$ s.t. $\forall$ list of orbit segments $\{(x_i, t_i)\}_{i=1}^k \subset X \times [0, \infty)$ $\exists$ ε-shadowing τ-connecting orbit: $y \in X$, $\tau_i \in [0, \tau]$ s.t. for $T_j = \sum_{i=0}^{j-1} t_i + \tau_i$ we get $f_{T_j}(y) \in B_{T_j}(x_j, \varepsilon)$ $\forall 1 \leq j \leq k$.

$B_t(x, \varepsilon)$ denotes the Bowen ball $\{y : d(f_s y, f_s x) < \varepsilon \ \forall 0 \leq s \leq t\}$

Expansivity + specification + regularity ⇒ uniqueness

Anosov flows are expansive: $\exists\varepsilon > 0$ s.t. "bi-infinite Bowen ball" $\Gamma_\varepsilon(x) = \{y : d(f_t y, f_t x) \leq \varepsilon \ \forall t \in \mathbb{R}\}$ contained in orbit of x.

Every Hölder potential φ for an Anosov flow has Bowen property:

$$\sup_{x,T} \sup_{y \in B_T(x,\varepsilon)} \left| \int_0^T \varphi(f_t x)\,dt - \int_0^T \varphi(f_t y)\,dt \right| < \infty$$

Theorem (Bowen 1974/75, Franco 1977)

Let f_t be an **expansive** *flow on a compact metric space with the* **specification** *property. Then every φ with the* **Bowen property** *has a unique equilibrium state μ_φ. Also, μ_φ has Gibbs property:*

$$\exists Q > 0 \ s.t. \qquad Q^{-1} \leq \frac{\mu_\varphi(B_T(x,\varepsilon))}{e^{-P(\varphi)T + \int_0^T \varphi(f_t x)\,dt}} \leq Q \qquad \forall x, T$$

Get ergodicity, partial mixing; for diffeos get K (Ledrappier 1977).

Countable Markov partitions ⇒ Bernoulli. Uniqueness?

Theorem (Ledrappier, Lima, Sarig 2016)

If $\dim M = 2$*, curvature* ≤ 0*,* $\varphi\colon T^1 M \to \mathbb{R}$ *is Hölder or* $q\varphi^{\text{geo}}$*, and* μ *is an eq. state for* φ *such that* $\mu(\text{Reg}) = 1$*, then* μ *is Bernoulli.*

Code as suspension over **ctbl-state** Markov shift (Lima, Sarig '17)

- Existence and uniqueness require extra information on the shift (Gurevich, Sarig), not available from Lima–Sarig result (but see Buzzi–Crovisier–Sarig for diffeos)
- Decay of correlations requires even stronger recurrence information (i.e. estimate on tail of Young tower)

For geodesic flows in nonpositive curvature, symbolic/Banach space approach does not (so far) say anything about existence, uniqueness, or correlation decay.

Unique MME from Patterson–Sullivan measure

The first uniqueness result in nonpositive curvature was...

Theorem (Knieper 1998)

There is a Patterson–Sullivan measure ν on the ideal boundary $\partial \tilde{M}$ s.t. the corresponding measure μ on $T^1 M$ is the unique MME. Then μ is fully supported and the limit distribution of per. orbits. As a corollary, there is an entropy gap: $h_{\text{top}}(\text{Sing}) < h_{\text{top}}(T^1 M)$

Theorem (Babillot 2002)

The product structure of μ leads to the mixing property.

For geodesic flows in nonpositive curvature, geometric Patterson–Sullivan–Knieper approach gives a unique MME but does not (yet) say anything about equilibrium states for $\varphi \neq 0$.

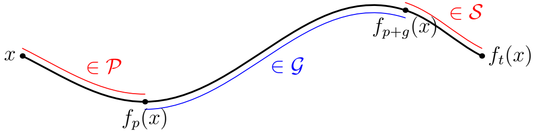
Decompositions of the space of orbit segments

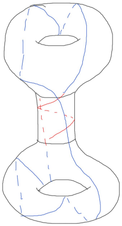
f_t a flow on a compact metric space X. A subset $\mathcal{G} \subset X \times [0, \infty)$ represents a collection of finite-length orbit segments.

$\mathcal{G}$ has specification if $\forall \varepsilon > 0\ \exists \tau$ s.t. every list of orbit segments $\{(x_i, t_i)\}_{i=1}^k \subset \mathcal{G}$ has an ε-shadowing τ-connecting orbit.

Same idea as before, but only needed for ***good*** *orbit segments*

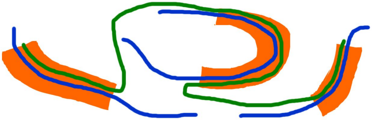
Decomposition: $\mathcal{P}, \mathcal{G}, \mathcal{S} \subset X \times [0, \infty)$ and functions $p, g, s \colon X \times [0, \infty) \to [0, \infty)$ s.t. $(p + g + s)(x, t) = t$ and $(x, p) \in \mathcal{P}$, $(f_p x, g) \in \mathcal{G}$, $(f_{p+g} x, s) \in \mathcal{S}$.





Obstructions to specification and regularity

Idea: $\mathcal{P}, \mathcal{S}$ are "obstructions to specification"; can glue **if** we first remove pre-/suffixes from $\mathcal{P}, \mathcal{S}$



Need obstructions to be "small"

Pressure of obstructions to specification:

- $Q_n = \{x \in X : (x,t) \in \mathcal{P} \cup \mathcal{S} \text{ for some } t \in [n, n+1]\}$
- $\mathbb{E}_n(\varepsilon) := \{E \subset Q_n : \forall x \neq y \in E \text{ we have } y \notin B_n(x, \varepsilon)\}$
- $\Lambda_n(\varphi, \varepsilon) := \sup\{\sum_{x \in E} e^{\int_0^n \varphi(f_t x)\, dt} : E \in \mathbb{E}_n(\varepsilon)\}$
- $P([\mathcal{P} \cup \mathcal{S}], \varphi) = \lim_{\varepsilon \to 0} \overline{\lim}_{n \to \infty} \frac{1}{n} \log \Lambda_n(\varphi, \varepsilon)$

Also require Bowen property for φ on $\mathcal{G}$ (not on all orbit segments)

$$\sup_{(x,T) \in \mathcal{G}} \sup_{y \in B_T(x,\varepsilon)} \left| \int_0^T \varphi(f_t y)\, dt - \int_0^T \varphi(f_t x)\, dt \right| < \infty$$

Small obstructions implies uniqueness

Pressure of obstructions to expansivity:

- $\Gamma_\varepsilon(x) = \{y \in X : d(f_t y, f_t x) \leq \varepsilon \ \forall t \in \mathbb{R}\}$
- If flow is expansive, then $\Gamma_\varepsilon(x) \subset$ orbit of x for all x
- $\mathrm{NE}(\varepsilon) = \{x \in X : \Gamma_\varepsilon(x) \not\subset \text{orbit of } x\}$
- $P^\perp_{\exp}(\varphi) = \lim_{\varepsilon\to 0} \sup\{h_\mu(f) + \int \varphi \, d\mu : \mu(\mathrm{NE}(\varepsilon)) = 1\}$

Theorem (C., Thompson 2016)

Suppose (X, f_t, φ) has $P^\perp_{\exp}(\varphi) < P(\varphi)$ and $\exists$ decomp $\mathcal{P}, \mathcal{G}, \mathcal{S}$ s.t.

1. *$\mathcal{G}$ has specification*
2. *φ has the Bowen property on $\mathcal{G}$*
3. *$P([\mathcal{P} \cup \mathcal{S}], \varphi) < P(\varphi)$*

Then (X, f_t, φ) has a unique equilibrium state μ. It is ergodic and has the Gibbs property on $\mathcal{G}$.

Decomposition for geod flow: first attempt, curvature of M

How to produce $\mathcal{P}, \mathcal{G}, \mathcal{S}$ for geodesic flow? Start with dim $M = 2$.

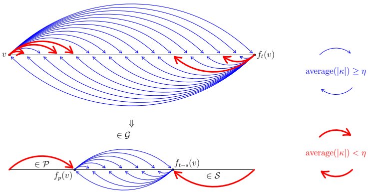
Idea: negative curvature $\rightsquigarrow$ hyperbolicity, so "obstructions" are

$$\mathcal{P} = \mathcal{S} = \mathcal{B}(\eta) := \{(v, T) : \int_0^T |\kappa(\gamma_v(t))| \, dt < \eta T\}$$

where $\kappa(x)$ is Gaussian curvature and $\eta > 0$ is a fixed parameter

Stripping away longest possible bad segments from ends leaves

$$\mathcal{G} = \{(v, T) : \int_0^t |\kappa(\gamma_v(s))| \, ds, \int_{T-t}^T |\kappa(\gamma_v(s))| \, ds \geq \eta t \ \forall t \in [0, T]\}$$



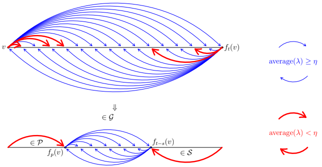
- Like hyperbolic times (Alves)
- What if dim $M > 2$? Then curvature is a tensor.
- Gromov example never has all sectional curvatures < 0

Decomposition: general solution, curvature of horospheres

Given $v \in T^1M$, let $H^s(v)$ be stable horosphere, $\mathcal{U}^s(v)$ its second fundamental form, and $\lambda^s(v) \geq 0$ the smallest eigenvalue of $\mathcal{U}^s(v)$. Similarly for $\lambda^u(v) \geq 0$, and then $\lambda = \min(\lambda^s, \lambda^u)$.

- $\lambda \colon T^1M \to [0, \infty)$ is a lower bound for curvature of horospheres, and thus bounds contraction/expansion rates

Fix $\eta > 0$ and let $\mathcal{P} = \mathcal{S} = \mathcal{B}$ be segments with average$(\lambda) < \eta$:



$$\mathcal{B} = \{(v, T) : \int_0^T \lambda(f_t v)\, dt < \eta T\}$$

$$\mathcal{G} = \{(v, T) : \int_0^t \lambda(f_s v)\, ds \geq \eta t, \int_{T-t}^T \lambda(f_s v)\, ds \geq \eta t \ \forall t \in [0, T]\}$$

Applying the general result: Sing controls obstructions

Claim: if $P(\text{Sing}, \varphi) < P(\varphi)$ then $\exists \eta > 0$ s.t. general result applies.

(1) $\mathcal{G}$ has specification

Holds $\forall \eta > 0$ by transitivity + local prod structure on Reg

(2) Bowen property on $\mathcal{G}$

Standard argument if φ Hölder.
**Different argument for φ^{geo}.

(0) $P^{\perp}_{\exp}(\varphi) < P(\varphi)$

$\text{NE}(\varepsilon) \subset \text{Sing}$, $P^{\perp}_{\exp} \leq P(\text{Sing})$

(3) $P([\mathcal{B}(\eta)], \varphi) < P(\varphi)$

$\mathcal{M}(\mathcal{B}(\eta)) \subset \mathcal{M}_\eta = \{\mu : \int \lambda \leq \eta\}$
$\therefore \bigcap_\eta \mathcal{M}(\mathcal{B}) \subset \bigcap \mathcal{M}_\eta = \mathcal{M}(\text{Sing})$,
so $\lim_{\eta \to 0} P([\mathcal{B}], \varphi) = P(\text{Sing}, \varphi)$

Theorem (Burns, C., Fisher, Thompson 2017)

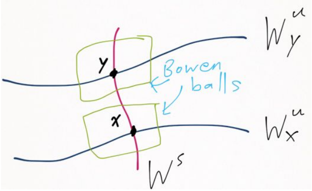
If $\varphi \colon T^1 M \to \mathbb{R}$ is continuous and locally constant on a neighbourhood of Sing, then $P(\text{Sing}, \varphi) < P(\varphi)$.

Ergodic properties: Gibbs ⇒ product structure ⇒ mixing

What about mixing, K, Bernoulli, decay of correlations?

- Our result only gives ergodicity and $\mathcal{G}$-Gibbs
- Ledrappier–Lima–Sarig gives Bernoulli if $\dim M = 2$
- No results (yet) on decay of correlations

For Anosov systems, Gibbs measures have product structure: use Gibbs property to control Radon–Nikodym derivative of holonomy maps between local unstable leaves



Can generalize this to our setting and use Pesin theory to prove that our measures μ_φ have quasi-product structure given in terms of $\partial\tilde{M}$ as with Patterson–Sullivan–Knieper. Then Babillot's machinery shows that μ_φ is mixing in any dimension.

A couple open questions

In dim 2, get gap for $q\varphi^{\mathrm{geo}}$ for all $q \in (-\infty, 1)$, since $\mu(\mathrm{Sing}) = 1$
$\Rightarrow h_\mu(f_1) = \int \varphi^{\mathrm{geo}}\, d\mu = 0$, so $P(\mathrm{Sing}, q\varphi^{\mathrm{geo}}) = 0 < P(q\varphi^{\mathrm{geo}})$

What about higher dimensions? May have $h_{\mathrm{top}}(\mathrm{Sing}) > 0\ldots$

If Sing = finite union of periodic orbits (e.g. analytic metric, dim 2) then for every Hölder $\varphi\colon T^1M \to \mathbb{R}$ there are φ_1 and φ_2 such that

1. φ and φ_1 are cohomologous, $\left(\varphi_1 = \frac{1}{T}\int_0^T \varphi \circ f_t\, dt\right)$
2. φ_1 and φ_2 are C^0-close,
3. φ_2 is locally constant on a nbhd of Sing.

Thus pressure gap is a C^0-dense (and open) condition.

Does the same result hold for the Gromov example?

thank you / merci