Equilibrium states and the ergodic theory of positive entropy surface diffeomorphisms

Bowen Legacy Conference

Omri Sarig

Weizmann Institute of Science

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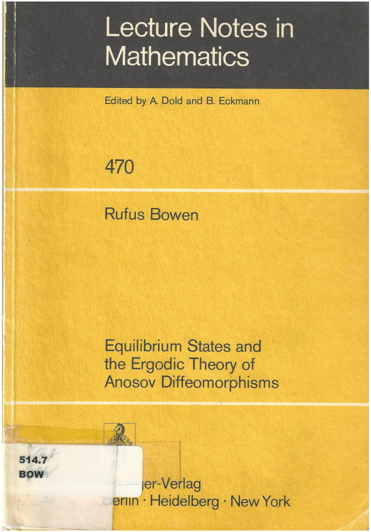

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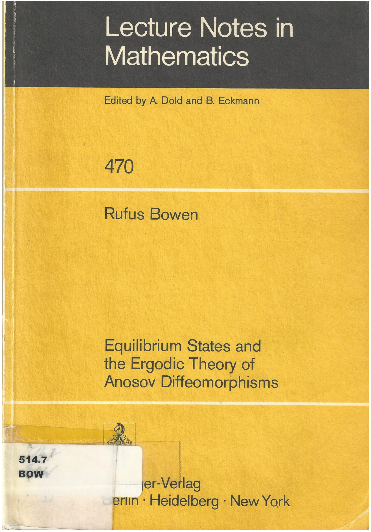
The Sinai-Ruelle program to uniformly hyperbolic diffeos.

Synopsis:

1. Coding by subshifts of finite type
2. Theory of equilibrium states for subshifts of finite type
3. Applications to Axiom A diffeos



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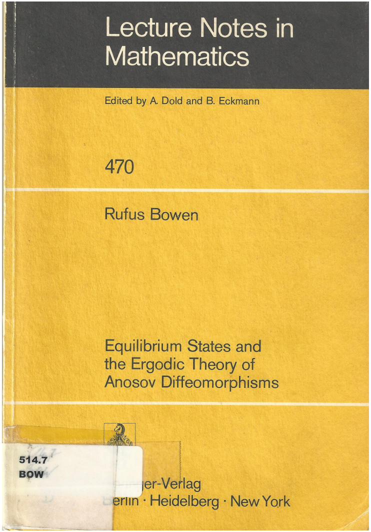

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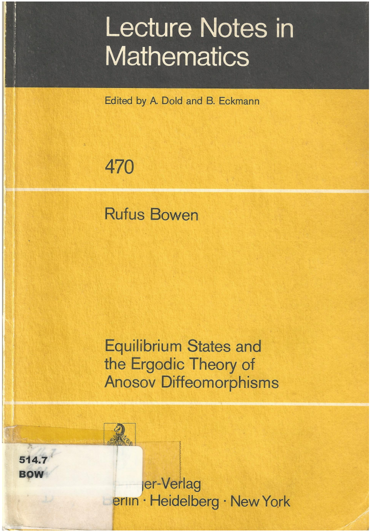

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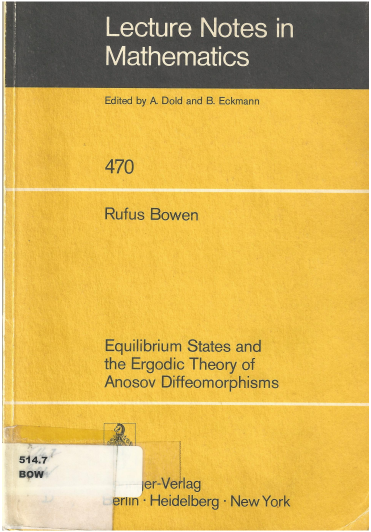

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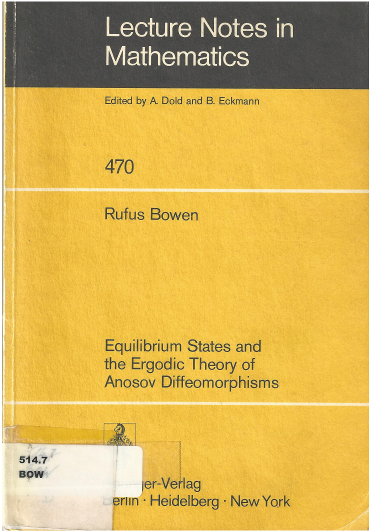

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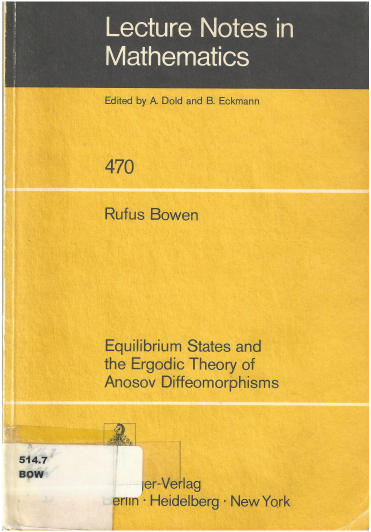

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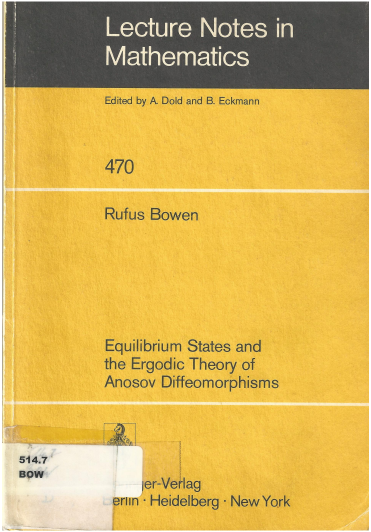

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Anosov/Axiom A diffeos	$\longrightarrow$	All positive entropy diffeos in dim 2 and other NUH systems in any dim

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Some Additions to the "tool box" since 1975:

Pesin theory (1976-7)

Ruelle's entropy inequality (1978)

Katok's horseshoe Theorem (1980)

Ledrappier-Young entropy theory (1985)

Yomdin theory (1987)

Katok-Strelcyn-Ledrappier-Przytycki theory for maps with singularities (1988)

Newhouse upper semi-continuity theorem (1989)

Aaronson-Denker-Urbanski theory of Gibbs-Markov maps (1993)

Young towers (1998)

This talk: advances on coding Non Unif Hyperbolic systems

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- NUH systems have countable Markov partitions
- Implications to the theory of equilibrium states
- (Some of) what we still do not understand

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Symbolic dynamics

Historical context

Markov Partitions before 1975:

- **First examples:** Smale codes the repeller of his horseshoe (1965); Adler & Weiss construct a MP for a Hyperbolic toral auto and use it to represent the Haar measure as a Markov measure (1967, 1970).
- **General theory:** Sinai defines and constructs MP for general Anosov diffeos (1968), and uses it to define "Gibbs measures" (1972). Bowen gives a new construction of MP for Axiom A diffeos (1970).
- **Flows:** Anosov (Ratner, 1973), Axiom A (Bowen, 1973)

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A completely new method for constructing MP, using pseudo-orbits and shadowing.

Anosov Shadowing Theorem

If f is a uniformly hyperbolic diffeo, then there are $\epsilon, \delta > 0$ s.t.

- If $\underline{x} = (x_i)_{i \in \mathbb{Z}}$ is an ϵ-pseudo orbit (i.e. $d(f(x_i), x_{i+1}) < \epsilon$ for all i)
- Then there is a unique $y := \pi(\underline{x})$ which δ-shadows $\underline{x}$ (i.e. $d(f^i(y), x_i) < \delta$ for all $i \in \mathbb{Z}$).

Bowen's idea (1975)

Fix a finite $\epsilon/2$-dense set of points S, then

1. $\Sigma := \{\epsilon\text{-pseudo-orbits in } S^{\mathbb{Z}}\}$ is a SFT, and
2. the shadowing map $\pi : \Sigma \to M$ satisfies $\pi \circ \text{shift} = f \circ \pi$
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Recent advances

A new Pesin-theoretic shadowing theory (S., 2013) allows to apply Bowen's strategy in the following setups:

1. All $C^{1+\epsilon}$ non-unif. hyperbolic diffeos (Ben-Ovadia, '17), e.g. all surface diffeos with ptv top entropy (S., '13).
2. All $C^{1+\epsilon}$ 3D flows with positive speed and positive topological entropy (Lima & S., '16)
3. Non-uniformly hyperbolic surface maps with singularities, e.g. billiards (Lima & Matheus, '17)

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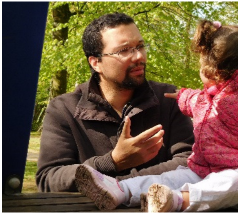


Precise statements

diffeos

Terminology

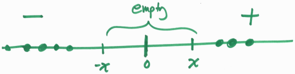
Countable Markov shift $\sigma : \Sigma \to \Sigma$

Let $G = G(V, E)$ be a countable directed graph, then we let

- $\Sigma = \{\text{all two-sided paths on } G\}$
- $\sigma[(v_i)_{i\in\mathbb{Z}}] = (v_{i+1})_{i\in\mathbb{Z}}$

χ-hyperbolic measure (for $\chi > 0$):

An ergodic invariant measure with Lyapunov spectrum



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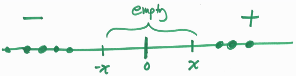

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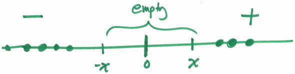

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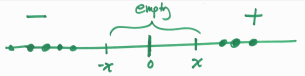

Setup

- M is compact smooth manifold w/o boundary
- $f : M \to M$ is a $C^{1+\epsilon}$ diffeomorphism

Theorem (Ben-Ovadia, '17; S., '13):

For every $\chi > 0$, $\exists$ locally compact countable Markov shift $\sigma : \Sigma \to \Sigma$ and a Hölder map $\pi : \Sigma \to M$ s.t.

- $\pi \circ \sigma = f \circ \pi$
- $\pi : \Sigma^{\#} \to M$ is finite-to-one*
- $\pi(\Sigma^{\#})$ has full measure for all χ-hyperbolic erg inv μ

Here $\Sigma^{\#} := \{\underline{x} \in \Sigma : (x_i)_{i<0}, (x_i)_{i>0}$ have constant subsequences$\}$, a set of universal full measure

* can be improved (Boyle-Buzzi).

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flows

Countable Markov Flow $\sigma_r : \Sigma_r \to \Sigma_r$

A suspension over a countable Markov shift with roof function which is Hölder cts, uniformly bounded away from $0, \infty$.

Setup

M is a compact smooth 3D manifold, and f^t a $C^{1+\epsilon}$ flow on M with positive top entropy and positive speed.

Theorem (Lima & S., '16):

For every ergodic hyperbolic measure μ, $\exists$ countable Markov flow $\sigma_r : \Sigma_r \to \Sigma_r$ and a Hölder $\pi : \Sigma_r \to M$ s.t.

- $\pi \circ \sigma_r^t = f^t \circ \pi$ for all t
- $\pi : \Sigma_r^\# \to M$ is finite-to-one
- $\pi(\Sigma_r^\#)$ has full μ-measure.

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A suspension over a countable Markov shift with roof function which is Hölder cts, uniformly bounded away from $0, \infty$.

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Examples of applications

Theorem (Lima & S. '16):

Suppose f^t is a 3D C^∞ flow with positive speed and positive topological entropy h, and let

$$\pi(T) := \#\{\text{closed orbits of length} \leq T\},$$

then $\pi(T) \geq const\left(\frac{e^{hT}}{T}\right)$ for all T large enough.

Related results

Margulis ('69), Parry & Pollicott ('83): for mixing Anosov/Axiom A flows, $\pi(T) \sim e^{hT}/hT$.

Katok ('80): For general NUH flows, $\liminf \frac{1}{T} \log \pi(T) \geq h$

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Under the above assumptions, every ergodic measure of maximal entropy is Bernoulli, or Bernoulli flow$\times$rotational flow.

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Ornstein & Weiss ('73): Geodesic flows in const neg curv.
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General flows: ergodic component of smooth measure with ptv entropy (Pesin '77); ergodic component of the SRB measure (Ledrappier ('84)).

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implications for equilibrium states

- Bowen codes by finite state Markov shifts.
- We code by locally compact countable state Markov shifts.

How does this affect
the theory of equilibrium states?

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Heuristics

Table of Contents

These notes came out of a course given at the University of Minnesota and were revised while the author was on a Sloan Fellowship.

The thermodynamic analogy

Hyperbolic systems + natural measures	$\leftrightarrow$	1D lattice gas models + Gibbs (DLR) states

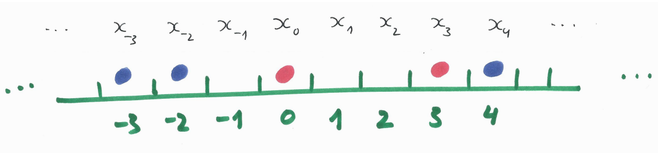
Y. Sinai: Gibbs measures in ergodic theory. Russian Math. Surv. 166 (1972), 21–64.

O. Lanford & D. Ruelle: Observables at infinity and states with short range correlations in statistical mechanics. CMP 13 (1969), 194–215.

D. Ruelle: A measure associated with Axiom A attractors. Amer. J. Math. 98 (1976), 619-654.

1D Lattice gas model

1D Lattice gas systems



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- **Interaction potential:** $U(x_0|x_1, x_2, x_3, \cdots)$
- **Gibbs state:** A probability measure on Σ which satisfies the Dobrushin Lanford Ruelle equations (ignoring temperature):

$$\mu(x_{-N}, \ldots, x_m | x_{m+1}, x_{m+2}, , \cdots) \propto \exp\left(-\sum_{k=0}^{N+m-1} U(\sigma^k x_{-N}^{\infty})\right) \quad \mu\text{−a.s.}$$

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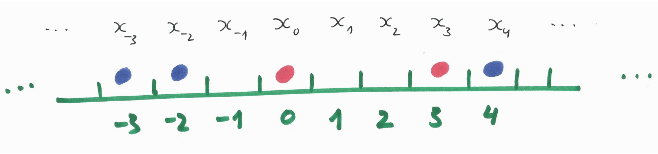

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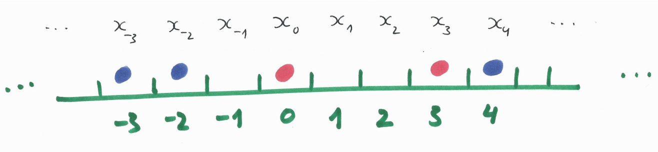

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Hyperbolic dynamical systems with "natural" invariant measures μ

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow[f]{} & M \end{array}$$

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Unif Hyperbolic + natural measures	$\to$	1D lattice gas + short range interactions	$\Rightarrow$	no phase transitions
NUH systems + natural measures	$\to$	1D lattice gas + long range interactions	$\Rightarrow$	???

Why long range?

- $\Sigma = \{$paths on an infinite directed graph $G\}$
- Σ locally compact $\overset{!}{\Rightarrow}$ every vertex of G has finite degree
  $\Rightarrow \text{diam}(G) = \infty$ with respect to the graph metric

$$\text{dist}_G(a, b) := \min\{n : a \text{ connects to } b \text{ in } n \text{ steps}\} + \min\{m : b \text{ connects to } a \text{ in } m \text{ steps}\}$$

Long range effect:

In any configuration $(\cdots, x_0, \cdots, x_n, \cdots)$, the value of x_n influences the value of x_0: $\text{dist}_G(x_0, x_n) \leq n$.

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Heuristics→Mathematics

Ruelle's operator

Setup

Σ=countable Markov shift, which is

- infinite alphabet
- topologically mixing (not necessary)
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- finite top entropy (not necessary)

Ruelle's Operator

- $\Sigma^+ = \{(x_0, x_1, \dots) : x \in \Sigma\}$
- $\phi = \phi(x_0, x_1, \dots)$ bounded, Hölder (can be relaxed), $\phi = -\beta U$
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Find a positive Radon measure ν, a ptv cts function h, and a ptv number λ s.t. $L_\phi h = \lambda h, L_\phi^* \nu = \lambda \nu$ with positive λ, h, ν.

Meaning:

- The eigenmeasure is a DLR state (on Σ^+)
- The eigenfunction is an invariant density
- The eigenvalue satisfies $\log \lambda = \sup\{h_\mu(\sigma) + \int \phi d\mu\}$
- The equilibrium measure is $h d\nu$ (if $\int h d\nu = 1$)

Lanford & Ruelle: Observables at infinity and states with short range correlations in statistical mechanics, CMP (1969)

D. Ruelle: A measure associated with Axiom A attractors, Amer. J. Math. (1976)

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So everything boils down to solving $L_\phi h = \lambda h, L_\phi^* \nu = \lambda \nu$

Finite state Markov shifts

A solution exists and is unique (Ruelle's Perron-Frobenius Theorem)

Countable state Markov shifts

There are two cases, which behave differently:

- Recurrent case: For some (all) $f \in C_c(\Sigma^+)$, $x \in \Sigma^+$, $\sum z^n (L_\phi^n f)(x) = \infty$ at the rad of convergence
- Transient case: For some (all) $f \in C_c(\Sigma^+)$, $x \in \Sigma^+$, $\sum z^n (L_\phi^n f)(x) < \infty$ at the rad of convergence

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Theorem (S., '01)

The recurrent case is exactly the case when $L_\phi h = \lambda h, L_\phi^* \nu = \lambda \nu$ can be solved, and ν is a conservative Radon measure. In this case h, ν are unique up to normalization.

Theorem (Van Cyr, '13)

In the transient case, and when Σ is locally compact, there exists a Radon measure ν s.t. $L_\phi^* \nu = \lambda \nu$.

Theorem (Ofer Shwartz, '17)

In the transient case, and when Σ is locally compact, there exists a positive continuous h s.t. $L_\phi h = \lambda h$. But in this case $h\nu$ is a dissipative infinite invariant measure. The solution is not necessarily unique $\rightarrow$Martin Boundary!

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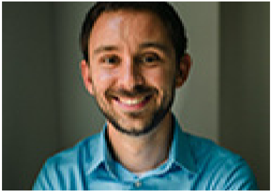

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Theorem (S. & Cyr '09; Cyr, '11):

For all locally compact* top. transitive countable Markov shift,

- There are locally Hölder transient potentials (Cyr);
- But they are rare: The set of locally Hölder recurrent potentials s.t. that L_ϕ acts on some rich Banach space with spectral gap is C^0-open, and Hölder dense. (Cyr & S.)

* Also for all non-locally compact shifts without the uniform Rome property (Cyr '11).

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- Are they lifts of Hölder functions on M?
- Newhouse-Yomdin Theory suggests that in the C^∞ case, the answer could be negative for potentials satisfying the Denker condition $\sup \phi - \inf \phi < P_{top}(\phi)$.

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Bowen's work is as relevant and fresh today as it was 40 years ago!