Current Trends in Dynamical Systems
and the Mathematical Legacy of Rufus Bowen

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Natural Invariant Measures

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Outline of talk

I. SRB measures -- from Axiom A to general diffeomorphisms

II. Conditions for existence and statistical properties of SRB measures

III. A class of "strange attractors" and some concrete examples

IV. Extending the scope of previous work, to infinite dim, random etc.

Part I. SRB measures: from Axiom A to general D.S.

SRB measures for Axiom A attractors (1970s)

M = cpct Riem manifold, f = map or f_t = flow

Assume uniformly hyperbolic or Axiom A attractor

A very important discovery of Sinai, Ruelle and Bowen is that these attractors have a special invariant prob meas μ with the following properties:

(1) (time avg = space avg)

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \to \int \varphi d\mu \quad \text{Leb-a.e. } x$$

for all cts observables

(2) (characteristic W^u geometry)

μ has conditional densities on unstable manifolds

(3) (entropy formula)

$$h_\mu(f) = \int \log|\det(Df|E^u)|d\mu$$

Moreover, (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3)

Proofs involves Markov partitions & connection to stat mech

Next drop Axiom A assumption.

How general is the idea of SRB measures ?

M = cpct Riem manifold, f = arbitrary diffeomorphsim or flow

Recall: properties of SRB measures in Axiom A setting:
(1) time avg = space avg, (2) characteristic W^u geometry, (3) entropy formula

Theorem [Ledrappier-Strelcyn, L, L-Young 1980s]

Let (f,μ) be given where μ is an arbitrary invariant Borel prob.
Then (2) $\Longleftrightarrow$ (3); more precisely:

(f,μ) has pos Lyap exp a.e. and μ has densities on W^u

$$\Longleftrightarrow \qquad h_\mu(f) = \int \sum_i \lambda_i^+ m_i \, d\mu$$

where λ_i are Lyap exp with multiplicities m_i

We defined SRB measures for general (f,μ) by (2).

Note: Entropy formula proved for volume-preserving diffeos (Pesin, 1970)

Entropy inequality ($\leq$) proved for all (f,μ) (Ruelle, 1970s)

What is the meaning of all this?

For finite dim dynamical systems, an often adopted point of view is

observable events = positive Leb measure sets

For Hamiltonian systems,

Liouville measure = *the* important invariant measure

Same for volume preserving dynamical systems

But what about "dissipative" systems, e.g., one with an attractor ?

Suppose $f : U \to U, \quad f(\bar{U}) \subset U, \quad \text{and} \quad \Lambda = \cap_{n=0}^{\infty} f^n(U)$

Assume f is volume decreasing.

Then $\text{Leb}(\Lambda) = 0$, and all inv meas are supported on Λ

i.e., no inv meas has a density wrt Leb

This does not necessary imply no inv meas can be ***physically relevant***

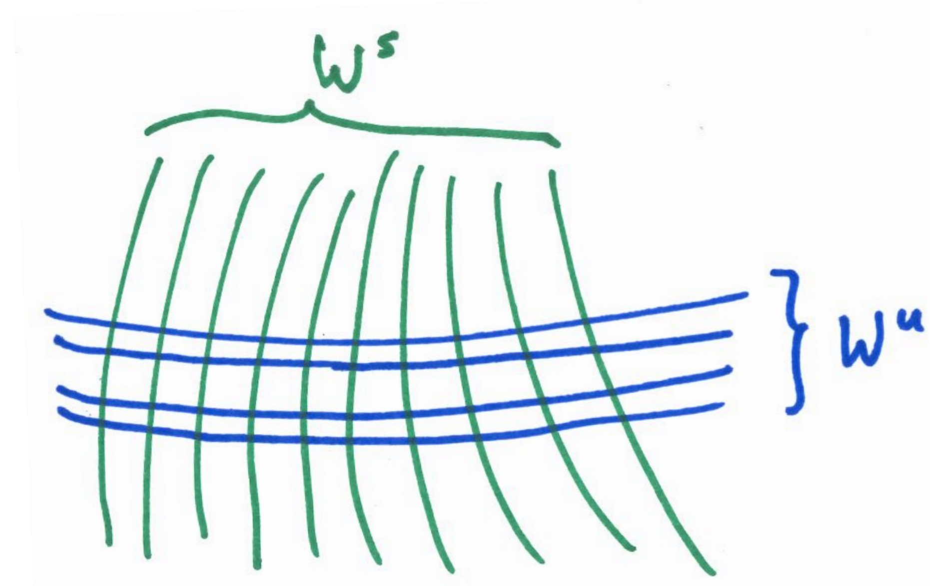
= reflecting the properties of Lebesgue

Here is how it works:

μ has densities on W^u together with

if $\bar{\varphi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_0^n \varphi(f^i x)$

then $\bar{\varphi}(x) = \bar{\varphi}(y) \quad \forall y \in W^s(x)$



$\Longrightarrow$

integrating out along W^s, properties on W^u passed to basin

Crucial to this argument is the absolute continuity of the W^s foliation

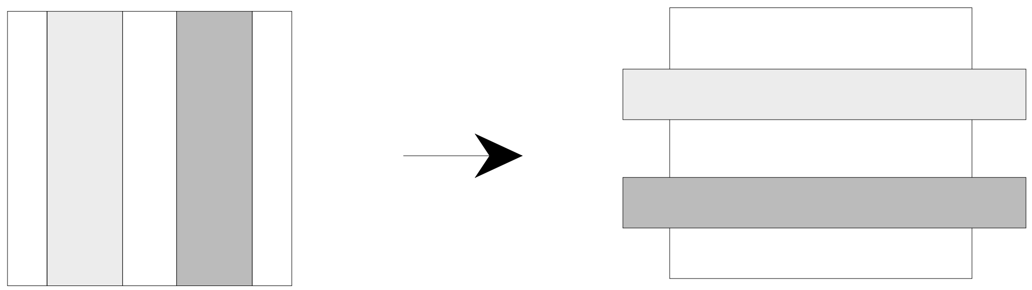
proved in nonuniform setting [Pugh-Shub 1990]

To summarize :

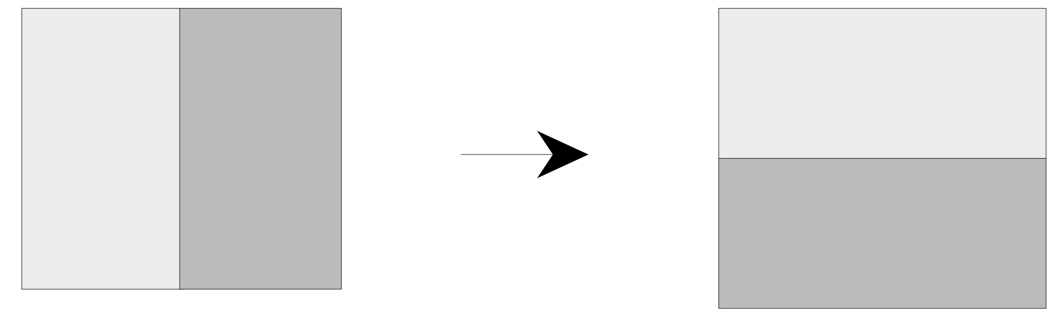
- one way to define ***SRB meas for general*** (f, μ) is
  pos Lyap exp + conditional densities on W^u i.e. property (2)

- conditional densities on W^u implies ***physical relevance***
  under assumptions of ergodicity and no 0 Lyap exp i.e. property (1)

And how is the entropy formula related to all this ?



entropy comes from expansion
but not all expansion goes into making entropy
Ruelle's entropy inequality



conservative case: no wasted expansion
Pesin's entropy formula

But whether entropy = sum of pos Lyap exp,
what does that have to do with backward-time dynamics?

Entropy formula holds iff system is conservative in forward time
= an interpretation of SRB measure

Meaning of gap in Ruelle's Inequality:

Theorem [Ledrappier-Young, 1980s] (f, μ) as above; assume ergodic for simplicity.
Then

$$h_\mu(f) = \sum_i \lambda_i^+ \delta_i$$

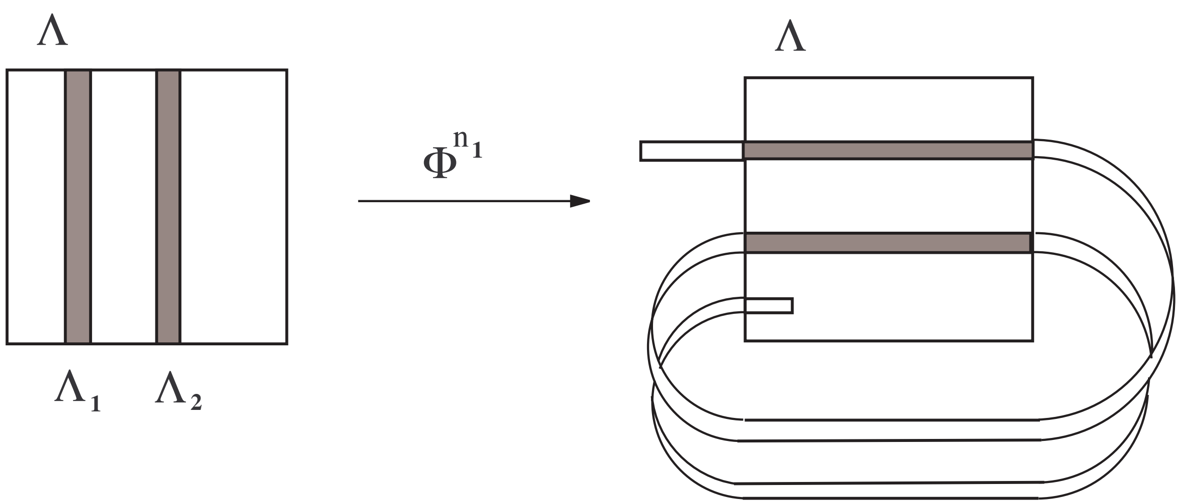
where $\delta_i \in [0, \dim E_i]$ is the dimension of μ "in the direction of E_i"

Interpretation: $\dim(\mu|W^u) = \sum \delta_i$ is a measure of disspativeness

Part II. SRB meas: conditions for existence & stat properties

A difference between results for Axiom A and general diffeos is that no existence is claimed

A natural condition that guarantees existence



Start with a reference box,
or a stack of W^u-leaves,
or a stack of surfaces roughly || to W^u
or just one such surface

Keep track of "good" returns to ref set
"good" = stretched all the way across
with unif bounded distortion

Let R = return time
m = Leb meas in E^u

Prop [Young 80s] If $\int R dm < \infty$, the SRB meas exists.

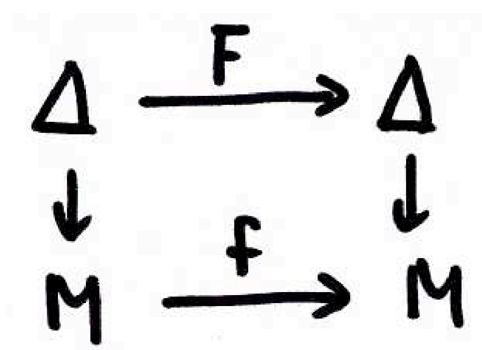
Most SRB meas (outside of Axiom A) were constructed this way.
First time I used it: piecewise unif hyperbolic maps of $\mathbb{R}^2$ [Young, 1980s]

In the same spirit that (finite) Markov partitions facilitated the study of statistical properties of Axiom A systems,
I proposed (1990s) that

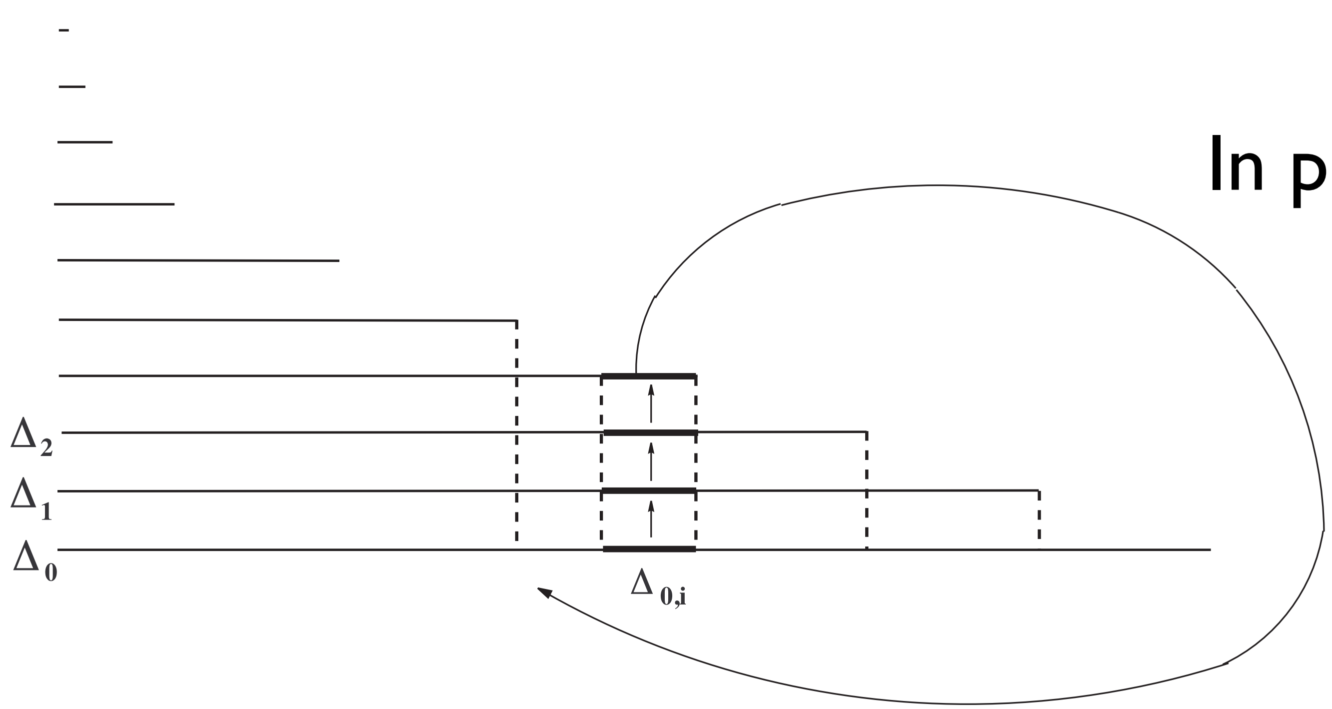
(1) stats of systems that admit ***countable Markov extensions*** can be expressed in terms of their renewal times, and

(2) this may provide ***a unified view*** of a class of nonuniformly hyperbolic systems that have "controlled hyperbolicity"

By (1), I mean given f, seek



s.t. (F, Δ) has a countable Markov partition



In practice, fix a reference set Λ_0 with hyperbolic (product) structure.
Build skyscraper until " good return "

"dynamical renewal"

Theorem [Young, 90s]
Suppose f admits a Markov extension with return time R, m=Leb,
(a) If $\int R dm < \infty$, then f has an SRB meas μ
(b) If $m\{R > n\} < C\theta^n, \theta < 1,$ then (f, μ) has exp decay of correl
(c) If $m\{R > n\} = O(n^{-\alpha}), \alpha > 1,$ then decay $\sim n^{-\alpha+1}$
(d) If $m\{R > n\} = O(n^{-\alpha}), \alpha > 2,$ then CLT holds.

Idea is to *swap messy dynamics for a nice space w/ Markov structures*

Construction of Markov extension was carried out for several known examples e.g.

Theorem [Young, 1990s] Exponential decay of time correlations for collision map of 2D periodic Lorentz gas

Remarks 1. Important progress in hyperbolic theory is the understanding that ***deterministic chaotic systems*** produce stats very similar to those from (random) ***stochastic processes***

2. Above are conditions for natural inv meas & their statistical properties.
To check these conditions, ***need some degree of hyperbolicity*** for the dyn sys

Part III. Proving positivity of Lyap exp in systems w/out inv cones

Major challenge even when there is a lot of expansion

Reason : where there is expansion, there is also contraction ....

v_0 = tangent vector at x, $v_n = Df_x^n(v_0)$

$\|v_n\|$ sometimes grows, sometimes shrinks

cancellation can be delicate

A breakthrough, and an important paradigm:

Theorems

$$f_a(x) = 1 - ax^2, \quad a \in [0, 2]$$

1. [Jakobson 1981] There is a positive meas set of a for which f_a has an invariant density and a pos Lyap exp.

2. [Lyubich; Graczyk-Swiatek 1990s] Parameter space $[0,2] = \mathcal{A} \cup \mathcal{B} \mod \text{Leb } 0$
   s.t. $\mathcal{A}$ is open and dense and $a \in \mathcal{A} \Longrightarrow f_a$ has sinks
   $\mathcal{B}$ has positive meas and $a \in \mathcal{B} \Longrightarrow f_a$ has acim & pos exp

Intermingling of opposite dynamical types makes it impossible to determine pos Lyap exp from finite precision or finite # iterates

Next breakthrough: The Henon maps [Benedicks-Carleson 1990]

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \qquad T_{a,b}(x,y) = (1 - ax^2 + y, bx)$$

[BC] devised (i) an inductive algorithm to identify a "critical set", and
(ii) a scheme to keep track of derivative growth for points that do not approach the "critical set" faster than exponentially

Borrowing [BC]'s techniques:

Theorem [Wang-Young 2000s] [technical details omitted]

Setting : $F_{a,\varepsilon} : M \circlearrowleft$ where $M = \mathbb{S}^1 \times D_m$ (m-dim disk)
a = parameter, ε^m = "determinant" (dissipation)

Assume

1. singular limit defined, i.e. $F_{a,\varepsilon} \to F_{a,0}$ as $\varepsilon \to 0$
2. $f_a = F_{a,0}|(\mathbb{S}^1 \times \{0\}) : \mathbb{S}^1 \circlearrowleft$ has "enough expansion"
3. nondegeneracy + transversality conditions

Then for all suff small $\varepsilon > 0$, $\exists\, \Delta(\varepsilon)$ = pos meas set of a

s.t. (a) $F_{a,\varepsilon}$ has an ergodic SRB measure
(b) $\lambda_{\max} > 0$ Leb-a.e. in M

We called the resulting attractors "rank-one attractors"
= 1-D instability, strong codim 1 contraction

- "fattening up expanding circle maps e.g. $z \mapsto z^2$ gives solenoid maps;
  slight "fattening up" of 1D maps (w/ singularities) gives rank-one maps

- passage to singular limit = lower dim'l object makes problem tractable

- rank-one attractors (generalization of Henon attractors) are currently
  the only class of nonuniformly hyperbolic attractors amenable to analysis

- proof in [BC] is computational, using formula of Henon maps;
  [WY]'s formulation + proof are geometric, independent of [BC]

- Motivation: rank-one attractors likely occur naturally, shortly after
  a system's loss of stability

- [WY] gives checkable conditions so results can be applied
  without going thru 100+ page proof each time

Application of rank-one attractor ideas

Shear-induced chaos in periodically kicked oscillators

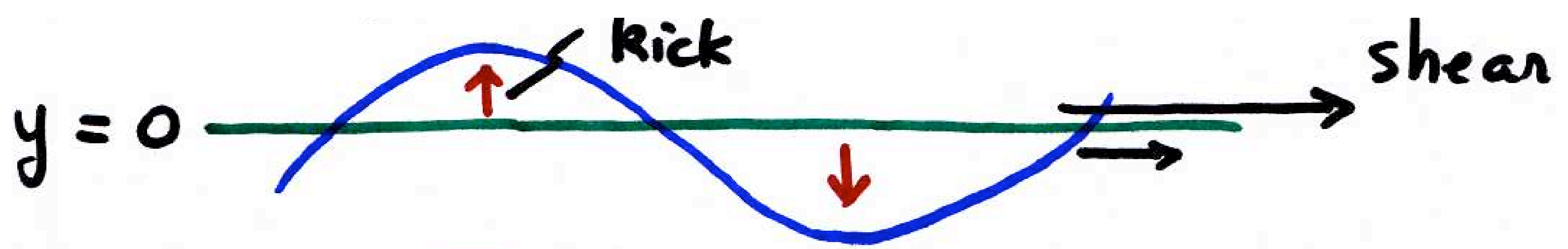
Simplest version: linear shear flow

kicks delivered with period T

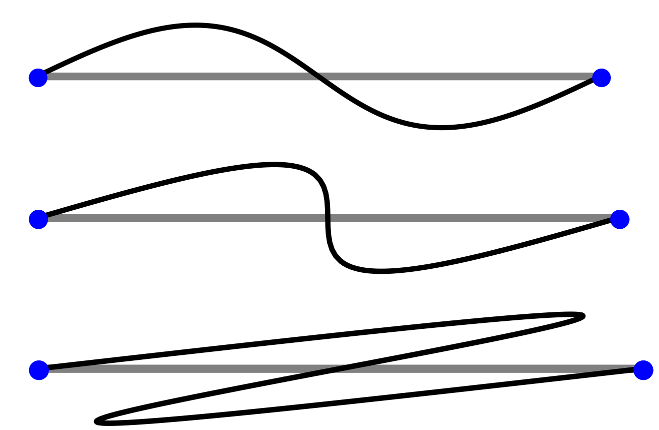
$$\frac{d\theta}{dt} = 1 + \sigma y$$
$$\frac{dy}{dt} = -\lambda y \qquad + A\sin(2\pi\theta)\sum_{n=0}^{\infty}\delta(t - nT)$$

$$\theta \in \mathbb{S}^1 \;,\; y \in \mathbb{R} \;,\quad \sigma = \text{shear} \;,\quad \lambda = \text{damping} \;,\quad T \gg 1$$

Unforced equation: $\{y = 0\}$ = attractive limit cycle



Increasing shear



key :

$$\frac{\sigma}{\lambda} \cdot A \;=\; \frac{\text{shear}}{\text{damping}} \cdot \text{deformation}$$

assuming $e^{-\lambda T} \ll 1$

$\dot{\theta} = 1 + \sigma y$ $\quad T$ = kick period $\quad \frac{\sigma}{\lambda} \cdot A = \frac{\text{shear}}{\text{damping}} \cdot \text{deformation}$

$\dot{y} = -\lambda y +$ kick $\quad F_T$ = time-T map of driven system

Theorem [Wang-Young 2000s]

(a) small $\frac{\sigma}{\lambda} A$: invariant closed curve

(b) as $\frac{\sigma}{\lambda} A$ increases : invariant curve breaks, horseshoes develop

(cf KAM)

(c) large $\frac{\sigma}{\lambda} A$: "dichomtomy"

ergodic SRB meas	$\lambda_{\max} < 0$ horseshoes + sinks
$\lambda_{\max} > 0$ pos meas set parameters	open set of parameters

Proof obtained by checking conditions in [WY]; general limit cycles OK.

Other applications of this body of ideas

- homoclinic bifurcations [Mora-Viana 1990s]
- periodically forced Hopf bifurcations [Wang-Young 2000s]
- forced relaxation oscillators [Guckenheimer-Weschelberger-Young 2000s]
- Shilnikov homoclinic loops [Ott and Wang 2010s]
- forced Hopf bif in parabolic PDEs, appl to chemical networks [Lu-Wang-Young 2010s]

Part IV. Extending the scope of existing theory

A. Infinite dimensional systems

Dynamical setting for certain classes of PDEs

Consider $$\frac{du}{dt} + Au = f(u)$$

where $u \in X =$ function space, $A =$ linear operator, $f =$ nonlinear term

To define a C^r dynamical system, need $(X, \|\cdot\|)$ s.t.

(1) $u_0 \in X \implies u(t)$ exists and is unique in X for all $t \geq 0$,

so semiflow $f^t : X \to X$ is well defined

(2) $t \mapsto u(t)$ is continuous for $t \geq 0$

(3) $f^t \in C^r$ for each t

This imposes restriction on the choice of $(X, \|\cdot\|)$

Remark. Dissipative PDEs (e.g. reaction diffusion eqtns) have attractors w/ a very finite dimensional character -- natural place to start

e.g. Multiplicative Ergodic Theorem proved only for Hilbert/Banach space operators that are quasi-compact [Mane, Ruelle, Thieullen, Lian-Lu ....]

In infinite dimensions: what plays the role of Leb measure ?

More concretely, what is a ``typical" solution for a PDE ?

Sample results

Theorem Under global invariant cones conditions:

(a) Existence of center manifold W^c — Constantin-Fois-Nicolaenko 80s, Chow, Sell, Mallet-Paret, Lu …

(b) Existence of stable W^s foliation [known]

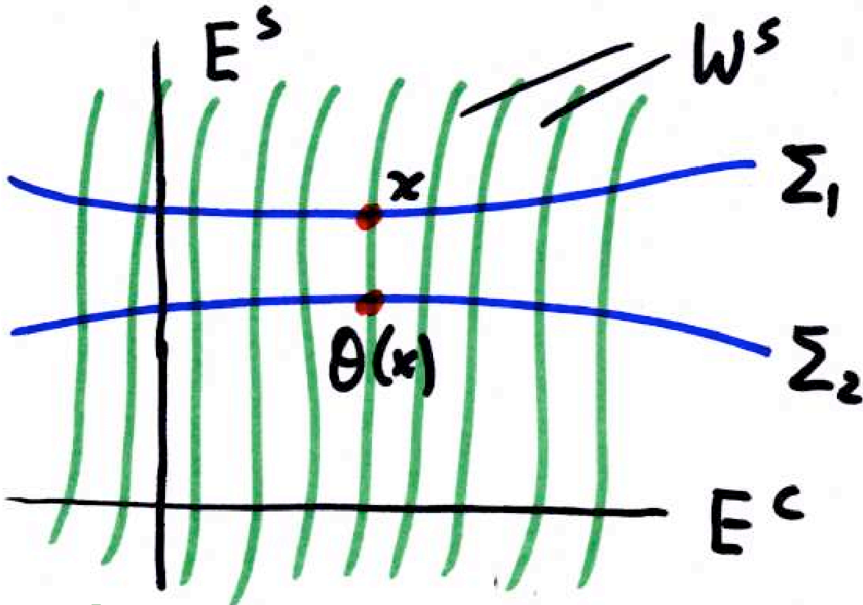
(c) Absolute continuity of W^s-foliation in the case $\dim(W^c) < \infty$ — [Lian-Young-Zeng 2010s]

i.e. if $\Sigma_1, \Sigma_2 =$ disks transversal to W^s,

and $\theta : \Sigma_1 \to \Sigma_2$ is holonomy along W^s-leaves,

then $\mathrm{Leb}(\theta(A)) \leq c\, \mathrm{Leb}(A)$ for all Borel $A \subset \Sigma_1$.

Interpretation



Notion of *"almost everywhere"* in Banach space inherited from Leb measure class on W^c

e.g. a.e. in the sense of k-parameters of initial conditions

General idea: use of *finite dim'l probes in infinite dim sp*

More general setting $X =$ Banach or Hilbert space

$F : [0,\infty) \times X \to X$ cts semiflow , $f^t(x) = F(t,x)$

Assume (1) $F|_{(0,\infty)\times X}$ is C^2

(2) f^t , Df^t_x injective [backward uniqueness]

(3) existence of compact $A \subset X$, $f^t(A) = A$ [attractor]

Theorem Assume no 0 Lyap exponents.

(a) [Li-Shu, Blumenthal-Young 2010s] μ is an SRB measure if and only if

$$h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \, d\mu$$

(b) [Blumenthal-Young 2010s] Absolute continuity of W^s

i.e. statistics of SRB ***visible***

B. Random dynamical systems (RDS)

$$\cdots f_{\omega_3} \circ f_{\omega_2} \circ f_{\omega_1}, \qquad i.i.d. \quad \text{with law } \nu$$

where ν is a Borel probability on $C^r(M)$= space of self-maps of

Motivation : small random perturbations of deterministic maps, **SDEs**

Two notions of invariant measures

Stationary measure $\mu(A) = \int P(A|x) d\mu(x)$

Equivalently, $\mu = \int (f_\omega)_* \mu \, \mathbb{P}(d\omega)$ in the random maps representation

Sample measures = μ conditioned on the past

$$\underline{\omega} = (\omega_n)_{n=-\infty}^{\infty} \qquad \mu_{\underline{\omega}} = \lim_{n\to\infty} (f_{\omega_{-1}} \circ \cdots \circ f_{\omega_{-n+1}} \circ f_{\omega_{-n}})_* \mu$$

Interpretation : $\mu_{\underline{\omega}}$ describes what we see at time 0 given that the transformations $f_{\omega_n}, n \leq 0,$ have occurred.

Theorem. Given RDS with stationary μ , $\lambda_{\max}$ = largest Lyap exp

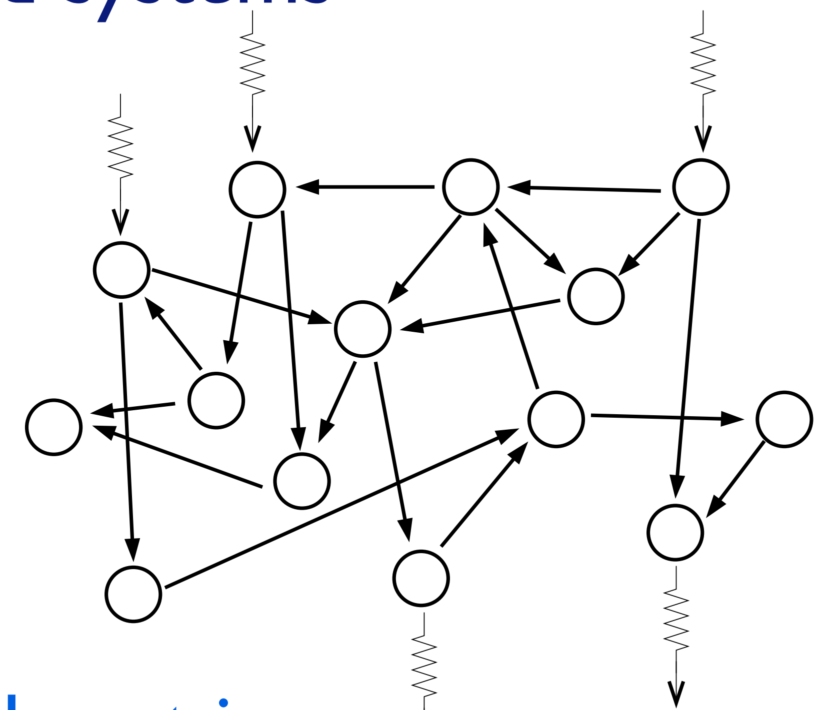
(a) [Le Jan, 1980s] If $\lambda_{\max} < 0,$ then $\mu_{\underline{\omega}}$ is supported on a finite set of points for $\nu^{\mathbb{Z}} - a.e. \underline{\omega}$ called ***random sinks*** .

(b) [Ledrappier-Young, 1980s] If μ has a density and $\lambda_{\max} > 0$, then entropy formula holds and $\mu_{\underline{\omega}}$ are ***random SRB measures*** for $\nu^{\mathbb{Z}} - a.e. \underline{\omega}$

(c) [L-Y 1980s] Additional Hormander condition on derivative process partial dimensions satisfy $\delta_i = 1 \quad \text{for} \quad i < i_0, \qquad \delta_i = 0 \quad \text{for} \quad i > i_0$

Application: ***reliability*** of biological and engineered systems

$I_\omega(t)$ → fluctuating input → (large) dynam sys → $R(t)$ → response



x_0 = is internal state of system at time of presentation

Say the dyn sys is ***reliable*** wrt a class of signals

if the dependence of $R(t)$ on x_0 tends to 0 with as t increases.

If $\lambda_{\max} < 0$ and a.e. $\mu_{\underline{\omega}}$ is, e.g. supported on a single point, then

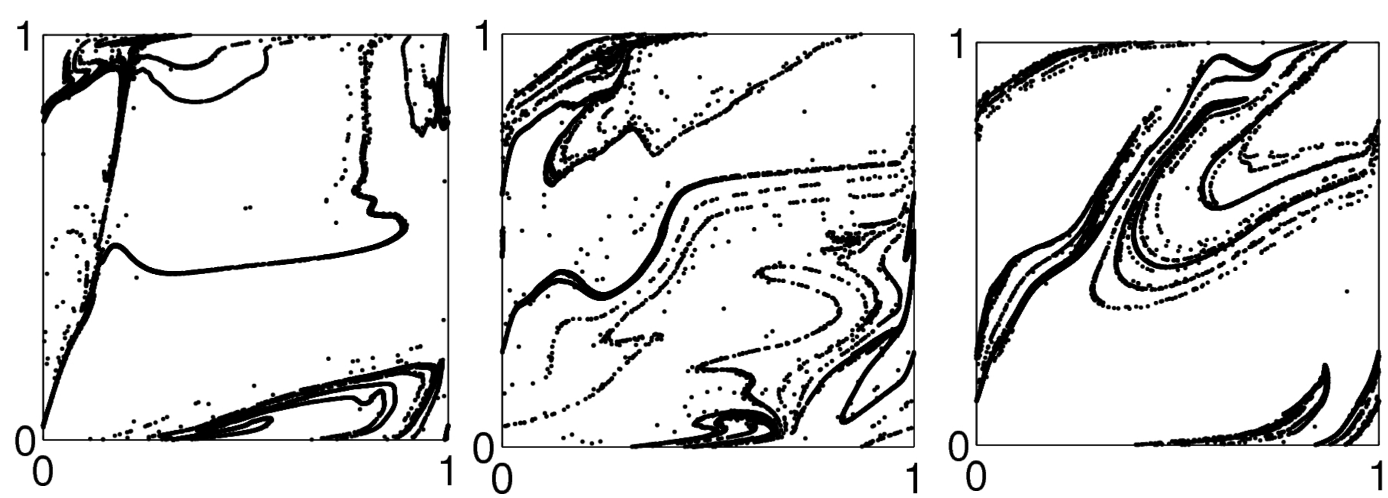
x_t = state of system at time t is largely indep of x_0 : ***reliable***

If $\lambda_{\max} > 0$ then $\mu_{\underline{\omega}}$ is supported on stacks of lower dim'l surfaces,

x_t depends on x_0 no matter how long we wait: ***unreliable***

Example: coupled oscillators

at $t = 50, 500, 2000$



Another application of RDS : **climate** e.g. Ghil group

stationary meas: theoretical avg vs sample meas: now given history

C. Open dynamical systems

= systems in contact with external world (rel to nonequilibrium stat mech)

A simple situation is leaky systems, i.e., systems with holes

Questions include escape rate ρ , surviving distributions etc

Sample result :

THEOREM [Demers-Wright-Young 2000s] Billiard tables with holes

(1) escape rate ρ is well defined

(2) limiting surviving distribution μ_∞ well defined & conditionally invariant

$$f_*(\mu_\infty)|_{M\setminus H} = e^{-\rho}\mu_\infty$$

(3) characterized by SRB geometry and entropy formula $h - \sum \lambda_i^+ m_i = -\rho$

(4) tends to SRB measure as hole size goes to 0

Result extendable to systems admitting Markov extension

D. Farther afield

In biological systems, I've encountered the following challenges :

(1) ***Inverse problems*** : Given basic structure + ***outputs*** of system, deduce dynamics and (nonequilibrium) steady states

(2) Continuous adaptation to (changing) stimulus, & partial convergence to ***time-dependent steady states***

Concluding remarks

- Importance of idea measured by impact and how it shapes future development , SRB ideas truly lasting
- Dynamical systems has evolved since the 1970s, will remain ***fun*** , ***vibrant*** , and ***relevant*** as long as it continues to evolve .....