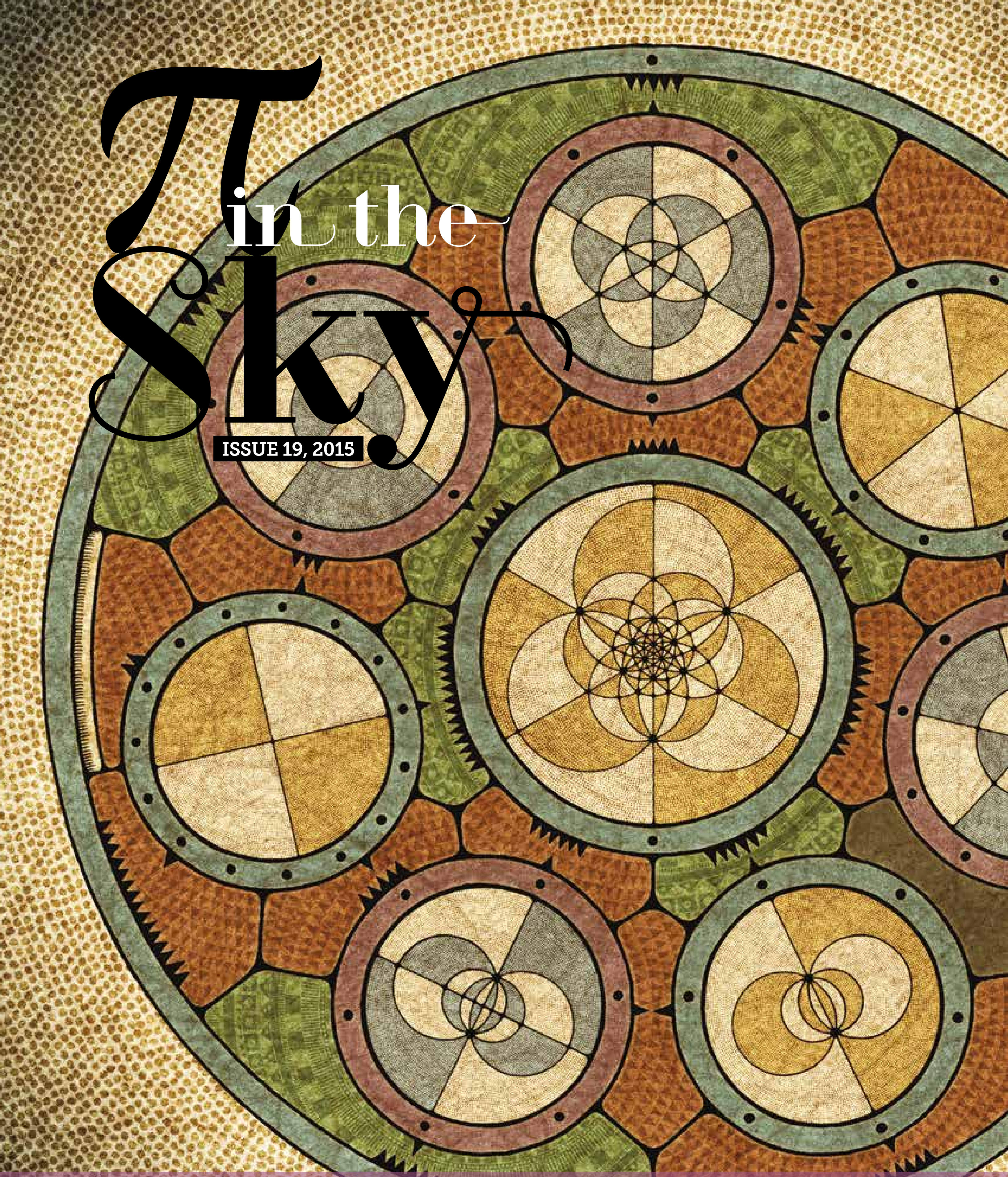
π in the Sky

ISSUE 19, 2015



Pacific Institute for the Mathematical Sciences

Pi in the Sky

Issue 19, 2015

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Pi in the Sky is aimed primarily at high school students and teachers, with the main goal of providing a cultural context/landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and articles may also put curriculum topics in a different perspective.

Welcome to Pi in the Sky!

The Pacific Institute for the Mathematical Sciences (PIMS) sponsors and coordinates a wide assortment of educational activities for the K-12 level, as well as for undergraduate and graduate students and members of underrepresented groups. PIMS is dedicated to **increasing public awareness** of the importance of mathematics in the world around us. We want young people to see that mathematics is a subject that opens doors to **more than just careers in science**. Many different and exciting fields in industry are eager to recruit people who are well prepared in this subject.

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Solutions to Math Challenges at the end of this issue will be published Pi in the Sky Issue 20. See details on page 26 for your chance to win $100!

For more information on our education programs, please contact one of our hardworking Education Coordinators.

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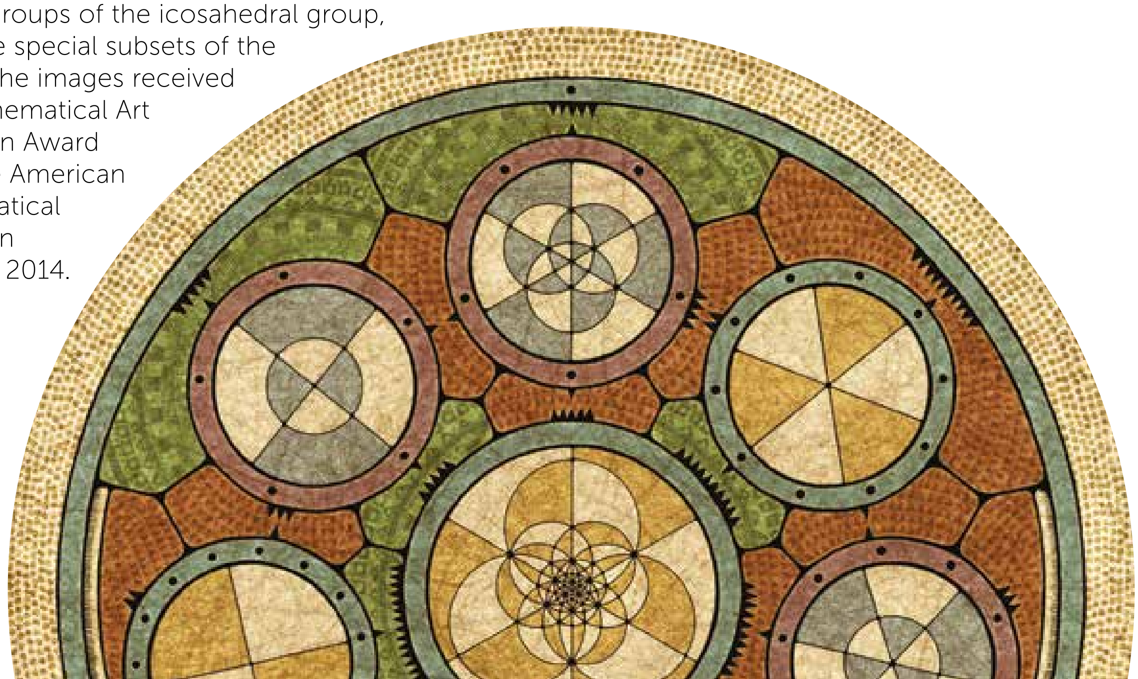
A Note on the Cover

The cover image, *Enigmatic Plan of Inclusion II*, is the second in a series of images by the artist, **Conan Chadbourne**, exploring the structure of the icosahedral group. That is, the collection of symmetries of an icosahedron. The panels depict the subgroups of the icosahedral group, i.e. some special subsets of the group. The images received the Mathematical Art Exhibition Award from the American Mathematical Society in January, 2014.

Conan Chadbourne holds a B.A. in Mathematics and Physics from New York University and lives in San Antonio, Texas, where he works as a freelance graphic designer and documentary film producer. www.conanchadbourne.com



EGYPTIAN MULTIPLICATION

AMRITA MITRA

Amrita Mitra is a computer science graduate and an enthusiast of mathematics. She has a hobby of reading and writing articles on historical mathematics.

IF I ASK YOU WHAT THE PRODUCT of 13 and 7 is, you will answer immediately: it is 91. And what if I ask you how got it? I am pretty sure you are going to laugh at me, because the question is too trivial.

$$\begin{array}{r} 13 \\ \underline{\times\ 7} \\ 91 \end{array}$$

Now, consider it is 2000 BC in Egypt and I ask you the same question. Will the answer of "how" be the same? It makes you think twice, doesn't it?

Ancient Egyptians were quite advanced in mathematics and knew how to multiply two numbers, but they used alternate methods of multiplication than we do today.

For example, to multiply two numbers such as 13 and 7, Egyptian scribes first identified which number to use as the multiplier and which number as the multiplicand. In this instance, let's use 13 as the multiplicand and 7 as the multiplier.

The next step is to write these numbers in columns, with the multiplier in the second column and doubled in each row. In the first column, write the numbers 1, 2, 4, 8, 16, 32, 64, … until you exceed the multiplicand. Here the multiplicand is 13, so we would write 1, 2, 4, 8, but not 16, because that would exceed the multiplicand 13.

So, the scribe would have written something like this:

1	7
2	14
4	28
8	56

Here is the twist: the Egyptians at that time were smart enough to know that in order to express any number from 1 to 2^n, we can add numbers from 2^0, 2^1, 2^2 … $2^{(n-1)}$. This is the system of binary arithmetic, which is used by computers.

The next step would be to identify the numbers from the left column that add up to the multiplicand (here 13).

Doubtful? Try it with any number! You can also prove it easily by mathematical induction if you use the strong induction hypothesis.

Here is an example: if a trader wants to give his customer any weight between 1 and 255 kg, what is the minimum number of weights he needs?

The answer is 8, and the weights are 1, 2, 4, 8, 16, 32, 64, 128. Using them, the trader can measure any weight between 1 to 255 kg. For example, to weigh 153 kg, the trader will use the following weights:

$$128 + 16 + 8 + 1 = 153.$$

You can try any other weights.

So, back to our multiplication question,

$$1 + 4 + 8 = 13.$$

To get the product of 13 and 7, the scribe would take 1st, 3rd and 4th row and add the numbers in the second columns of those rows. In our case, the numbers would be 7, 28 and 56. When we add those we get

$$7 + 28 + 56 = 91.$$

Bingo! We have indeed got the correct product of 13 and 7. Now, isn't that interesting?

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1. R. GILLINGS, Mathematics in the time of the Pharaohs, Dover Publications (1982).

2. http://www-history.mcs.st-andrews.ac.uk/Indexes/HistoryTopics.html.

A BASEBALL in a glass of BEER

Reproduced with permission: L. WEINSTEIN, *Guesstimation 2.0: Solving Today's Problems on the Back of a Napkin*, Princeton University Press (2012) 33-36.

WHILE I WAS AT A NORFOLK Tides baseball game, a foul ball landed in the section above me and showered some of my friends with beer. What is the probability of a foul ball landing in a cup of beer during one baseball game? What is the expected number of "splash downs" during all the major league baseball games played in an entire season? (See the answers for an even more improbable detail.)

Hint How many foul balls land in the stands each game?

Hint What is the size of a cup of beer?

Hint What fraction of people have beer cups?

Answer We need to break down the problem into manageable (or at least estimable) pieces. The first two pieces will be the number of foul balls per game that land in the stands and the probability that a given foul ball lands in a cup of beer. Let's start with the number of foul balls that land in the stands. The number per inning is definitely more than one and fewer than twenty, so we can take the geometric mean of five for our estimate.

[1]*If you watch a lot of baseball, you can probably come up with a better estimate; I counted several (between three and seven) foul balls per inning landing in the stands. With nine innings per game, this amounts to forty foul balls per game that could possibly land in a cup of beer.*

Now we need to estimate the probability that a given foul ball will land directly in a cup of beer. (Note: only beer is sold in open-topped cups.) This means that we need to break the problem into even smaller pieces. Let's assume that the cup of beer is sitting innocently in a cup holder. To hit a cup of beer, the foul ball needs to:

1. not be caught by a fan, **2.** land within the area of a seat, **3.** hit a seat whose owner has a cup of beer, **4.** land in the cup.

Most fly balls are caught, but many are not. Let's estimate that between one-quarter and one-half of fly balls are *not caught*. "Averaging" the two, we will use one-third.

1 Yes, I know the square root of 20 is a touch less than 4.5. If you prefer to round down and use four rather than five, go ahead.

Most of the stadium area is used for seating, so let's ignore that factor. At any given time, more than 1% and less than 100% of fans have a cup of beer in front of them. Using the geometric mean, we estimate that 10% of seats have beer cups.

A large beer cup is 4 inches (10cm) across, so the baseball must land in an area defined by

$$A_{cup} = \pi r^2 = 3(2in)^2 = 10in^2.$$

The area of the seat (from arm rest to arm rest and from row to row) is about 2 ft by 3 ft (60cm by 90cm), so

$$A_{seat} = (24\,in)\times(36in) = 10^3 in^2.$$

Thus, if the ball hits a seat that has a cup of beer, the probability that it lands in the cup is

$$P_{cup} = \frac{A_{cup}}{A_{seat}} = \frac{10in^2}{10^3 in^2} = 10^{-2}$$

or 1%. The metric probability is the same.

(*Extra credit question* Which is more likely, that the balls lands in the cup in the cup holder, splashing the beer, or that the fan is holding the cup of beer when the foul ball arrives and splashes it in his or her excitement?)

This means that the probability that any single foul ball lands in a cup of beer is

$$P = \frac{1}{3}\times\frac{1}{10}\times(10^{-2}) = 3\times10^{-4}.$$

With forty foul balls per game, this means that the probability of a foul landing in a cup of beer during any one game is 10^{-2} . This is not very likely. The probability that we will be directly below the splash is even less likely. During the entire season, each of the 30 teams plays 160 games, giving a total of about 2,000 games (as it takes two teams to play a game). This means that the total number of beer landings in one season is

$$B = (2 \times 10^3 \text{ games per season})$$

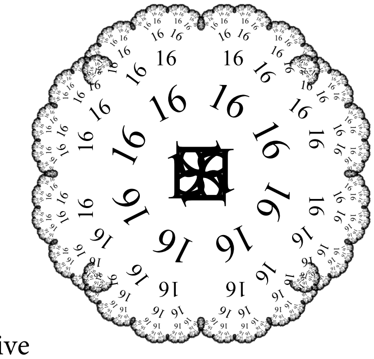
$$\times (10^{-2} \text{ beer landings per game}) = 20 .$$

Because baseball analysts keep meticulous statistics, I am very surprised that they do not appear to record beer landings.

Oh yes. The very improbable detail? According to my friends, the beer belonged to our former governor! ("Now at an improbability factor of a million to one against and falling," D. Adams[2].)

2 D. Adams. The Hitchhiker's Guide to the Galaxy, Pan Books, London, (1979).

Sweet Sixteen



MICHAEL P. LAMOUREUX, Professor of Mathematics, University of Calgary

WHAT'S SO SWEET ABOUT SIXTEEN?

Last summer, I was invited to give a talk at Calgary's 16th Pecha Kucha event. These events run around the world, where artists and other creative people get together and present 20 slides, in 20 seconds each, about some fun topic. That night's topic was "Sweet Sixteen" — that special time of coming-of-age, when a teenager is crossing the threshold into adulthood. We heard about dancers discovering their art, activists discovering the world of social change, even how to make the perfect cupcake!

Fig. 1. Computer output in base 16.

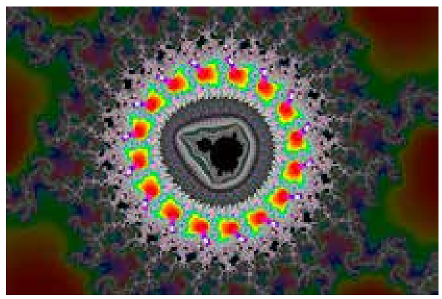


Fig. 2. 16 dots in the Mandelbrot fractal.

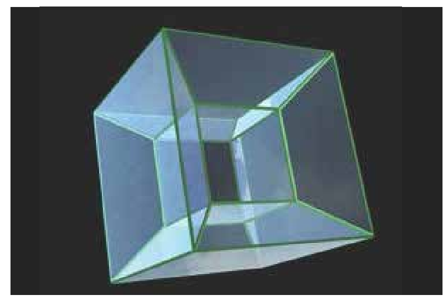


Fig. 3. 16 corners in a 4 dimensional cube.

As a mathematician, I thought I should talk about the number 16 itself.

There's a lot you can say about 16, as it is a very interesting number. In fact, all integers are interesting: here's a proof. Make a list of all the interesting integers (like 1, the first number, that's interesting, or 2, the first prime, that's interesting and so on.) Make another list of everything left over — the uninteresting integers. In this second list, there is a smallest number — the smallest uninteresting integer. Now THAT would be an interesting number! So that's a contradiction. Unless, of course, the list of uninteresting numbers was empty.

But for 16 itself — well, there's a lot you can say. We often see numbers in base 16 when working with computers, as in Figure 1. We see 16 appear in the strangest places, for instance in a pattern of 16 red dots buried in the Mandelbrot set, as in Figure 2. When we go into four dimensional space, we see 16 corners on a 4D cube, as sketched in Figure 4.

But here is something really neat about 16. It is the ***only*** integer that can be written two different ways as the power of the two same integers. That is, we can write

$$16 = 2^4 = 4^2.$$

16 is 2 raised to the power 4, and also 4 raised to the power 2. *Sweet!*

Now, you can play around with other pairs of integers and find many examples that don't work. For instance

$$8 = 2^3 \neq 3^2 = 9,$$

or

$$81 = 3^4 \neq 4^3 = 64.$$

Play around all you want, you won't find two integers where one, raised to the other, gives you the same result when you flip the numbers.

So, why is that? Well, if you have two integers x,y that satisfy

$$x^y = y^x,$$

taking logarithms we find that

$$y \log x = x \log y, \text{ and so } \frac{\log x}{x} = \frac{\log y}{y}.$$

A quick plot of $z = \log(x)/x$ in Figure 4 shows that for any z value in the range of the function, you can always find exactly one x to the left of the function's maximum and one y to the right of the max, with $\log(x)/x = \log(y)/y = z$. A little calculus shows the maximum is exactly at the point $(e,1/e)$, where $e \approx 2.71$ is the base exponential. So the x is in the interval $(1,e)$ and the y is in the interval (e,∞) while z runs between 0 and $1/e$.

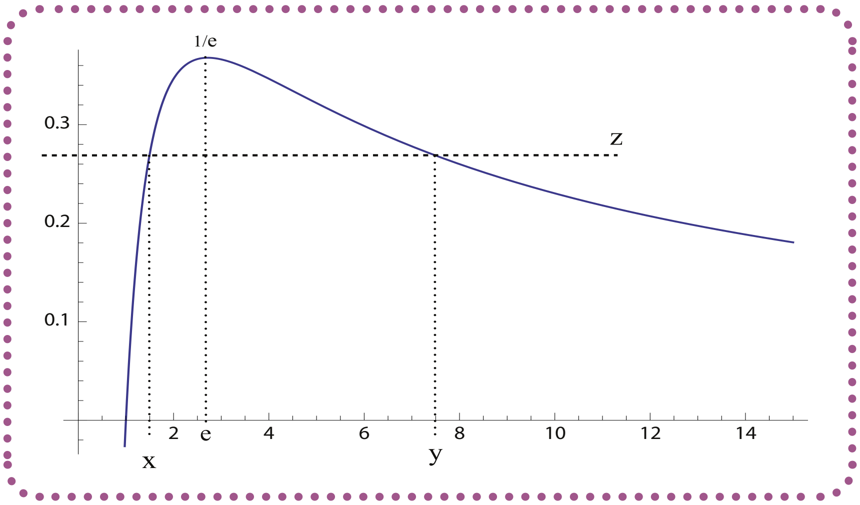


Fig. 4. A plot of log(x)/x.

Well, there is only one integer in the interval $(1,e)$, which is x=2. The corresponding y value is y =4, since we have

$$\frac{\log 2}{2} = \frac{\log 4}{4}.$$

Which brings us back to the number 16, expressed as two powers 2^4 and 4^2.

But now, Figure 4 suggests we can find a lot of pairs of numbers x,y with $x^y = y^x$, except they might not be integers. A little playing around with a calculator shows that we can write the integer 17 in two different ways as a power, with

$$17 = 1.78381425..^{4.89536796...} = 4.89536796..^{1.78381425...}.$$

(At least to the accuracy of my calculator.)

In fact, here is a theorem. For <u>any</u> number N bigger that $e^e \approx 15.1543$, we can find two different numbers x,y so that

$$N = x^y = y^x.$$

That is, we can write N in two different ways, x raised to the y, or y raised to the x.

Which again shows why 16 is an interesting integer: it is the smallest integer that can be written two different ways x^y and y^x, even allowing x and y to be <u>any</u> numbers (not just integers). 15 is too small, because it is smaller than e^e.

So, where does this e^e come from? You can see it in Figure 4. As you push z up towards the maximum $1/e$, the corresponding x and y move towards e. Thus the powers $x^y = y^x$ head towards e^e. That, it turns out, is the minimum value those paired powers can take. Conversely, if you move the z down towards zero, the x heads towards 1, while the y heads towards ∞. A careful calculation with limits shows the powers $x^y = y^x$ both head towards infinity. So the powers cover everything between e^e and ∞.

Can we say more? Amazingly, yes. This is an interesting function, the Lambert W function, which solves the equation

$$z = W\exp(W).$$

There are really two branches to this function, called $W_0(z)$ and $W_{-1}(z)$. With a bit of algebra (an exercise!) it is easy to show that starting with the value z shown on the vertical axis of Figure 4, we find

$$x = \frac{W_0(-z)}{-z} \text{ and } y = \frac{W_{-1}(-z)}{-z}.$$

So, pick z to be any number between 0 and $1/e$, plug into the above formulas, and you get two numbers x,y so that their powers match: $x^y = y^x$. With a good choice of z, you can find any power between e^e and infinity.

With all the math tools available on the internet, Wolfram Alpha, Mathematics, Google calculator and more, you can find these functions yourself and make your own computations of the x and y.

Have fun!

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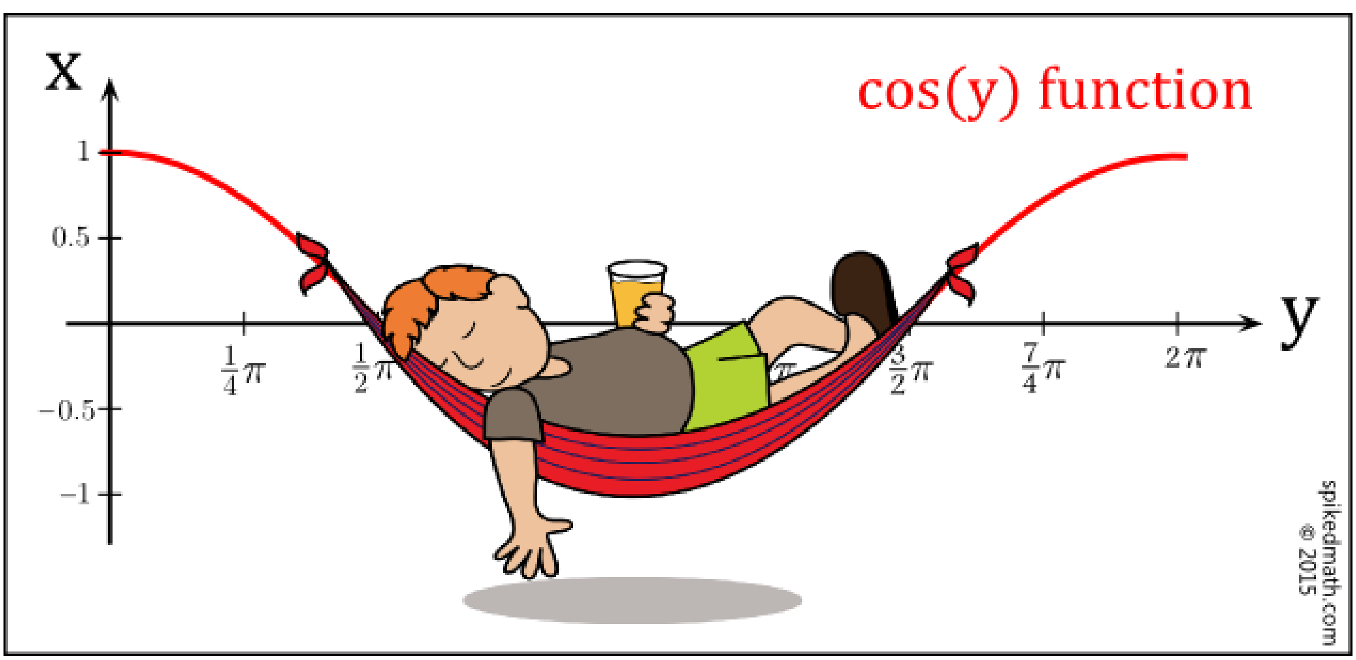
Google Calculator: www.google.com
Pecha Kucha: www.pechakucha.org/cities/calgary
Wolfram Alpha: www.wolframalpha.com



FORETELLING DEATH: THE MATH OF WAR

SOPHIA LUO is a senior at The Harker School, California. She enjoys creating with art and hiking with friends.

FOR MILLENNIA, HUMANS HAVE TRIED TO PREDICT THE FUTURE. FROM SHAMANS to witches, magicians and psychics, we are forever wondering what will happen next. Nowadays, we have new groups of people who cast prophecies about the future; economists use statistics to predict whether our financial markets will rise or fall and doctors use biology to foretell the consequences of diseases for human bodies. And now, for better or for worse, some of the latest research indicates that we can use math to predict wartime deaths.

A team of physicists, mathematicians, computer scientists and economists decided to verify this prediction by targeting recent Middle Eastern and Latin American conflicts, such as those in Iraq, Afghanistan and Columbia. Forming a research group, they aimed to observe the mathematical and statistical patterns in war – a chaotic and seemingly unpredictable human phenomenon.

To collect data they looked at 130 sources, all of which are accessible to the public. Examining sources ranging from newspapers to NGO reports and television news, the researchers extracted data concerning the size of attacks, the number of deaths, conflict locations and more. During "The Mathematics of War" TED talk, Sean Gourley, one of the researchers, observed, "All this noise around us actually has information." In other words, the team drew upon *big data* methods through their extraction and analysis of specific pieces of information from the large open-source expanse of the Internet.

With the data, the research team concluded that (1) the conventional theories regarding strict hierarchies and networks of insurgent groups may prove false, and (2) similar patterns between war and financial markets give rise to the possibility that collective human behavior in violent and nonviolent situations may be related. To reduce error, researchers individually synthesized information from each wartime event and verified the data by comparing results with separate groups of researchers. The team also compared different stories of the same events from various sources to minimize media and reporting bias. Finally, the researchers took into account only deaths as measurements of casualties, as injuries would be much more difficult to use as data points.

At the heart of their research paper, "Common ecology quantifies human insurgency" [1], the team created three summative figures (Power-Law Exponents and Size of Events [here, fig. 1 and 2], as well as Timing of Events) and a diagram (Model Framework for Insurgency) that revealed very startling patterns and behaviors in war. Together, these findings point to the possibility of quantifying the chaos of war. In order to better

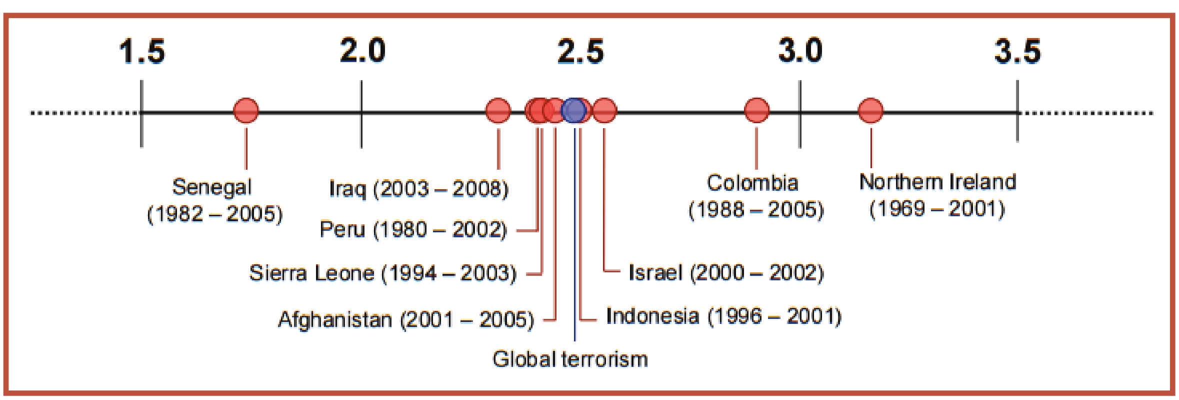


Fig. 1. Power-Law Exponents

understand these patterns, I will summarize the researchers' findings, including the mathematical theory and statistical analysis, behind the first figure: Power-Law Exponents.

For each war they assessed, the researchers graphed the relationship between the number of people killed in each attack (i.e., event casualties) and the frequency of those attacks. They found negative correlations between the two variables in all the wars. Applying a statistical procedure derived by Clauset, Shalizi and Newman in the paper "Power-Law Distributions in Empirical Data" [2], the researchers found a power-law relationship between the negative slopes of these frequency versus casualty plots and the probabilities of a certain number of deaths occurring due to an attack:

$$P(x) = Cx^{-\alpha}$$

where P represents the probability of x number of deaths, C is a constant and α stands for the slopes of frequency versus casualty plots. A power-law relationship is a statistical relationship in which one variable changes as an exponent of another. Moreover, because $P(x) \to \infty$ as $x \to 0$, there must be a lower bound, x_{min}.

After calculating α for all their selected wars, they found that most of the values clustered around 2.5, the value correlated with global terrorism. Figure 1 depicts all the found αs.

To find these numbers, they first needed to approximate α and x_{min}. For each war, they used event casualties to find respective complementary cumulative density functions (CCDF), defined as $P(x \geq x_{min})$:

$$\text{CCDF}(x) = P(x \geq x_{min})$$

For an example illustrating CCDF (let's define it as $\text{CCDF}(X) = P(X \geq x)$ for the following explanation), imagine a situation of flipping a fair coin twice. Here are all the possible results for tallying the number of heads flipped:

	Definition	Probabilites	Calculation
X = 2	2 heads (h,h)	1/4	$CCDF(2) = P(X \geq 2) = 1/4$
X = 1	1 head (t,h) or (h,t)	2/4	$CCDF(1) = P(X \geq 1) = 2/4 + 1/4 = 3/4$
X = 0	No heads (t,t)	1/4	$CCDF(0) = P(X \geq 0) = 1/4 + 3/4 = 4/4$

Table 1. All the possible results for tallying the number of heads flipped

The researchers applied this same mathematical reasoning to their study. Instead of flipping a coin, they calculated the power-law distributions of event sizes for each war. After finding possible x_{min} values, they estimated corresponding α values. Table 2 is a regenerated table of a portion of their data from their paper's Supplementary Information.

Having generated α values for each conflict, the researchers also provided visual demonstrations of the power-law relationships applied to four different wars in Afghanistan (2001-2005), Iraq (2003-2008), Colombia (1988-2005) and Peru (1980-2002), as shown in Figure 2.

On the log-log axes, the green curves are smooth lines drawn through the data and the blue lines represent power law relationships fitted to the data. Like other power law relationships drawn on log-log plots, they appear as straight lines.

Now the real question is, what if we applied the researchers' methods to predict the futures of wars being fought? After collecting a substantial number of data points, wouldn't we be able to calculate α values and fit relatively accurate power-law models to incipient wars?

Conflict Country	Database start-end dates	Total number of events	Total number of Deaths	Deaths explained by power-law estimate	Number of events with deaths $\geq X_{min}$	X_{min}	a
Colombia	01/01/1988-22/01/2005	21,478	38,876	20,637	2,251	4	2.90
Iraq	01/01/2003-24/11/2005 01/05/2003-21/07/2007 01/04/2007-01/09/2008	3,737 8,645 4,632	22,347 31,671 15,427	19,184 19,262 9,834	1,186 1,435 898	2 4 3	2.03 2.32 2.31
Senegal	10/06/1982-04/02/2005	559	3,313	3,199	191	2	1.73
Afghanistan	09/09/2001-29/10/2005	1,225	5,048	3,655	184	8	2.44
Israel-Palenstine	01/07/2000-31/07/2002	180	811	524	44	6	2.55
Sierra Leone	13/10/1994-10/01/2003	697	13,596	9,933	83	47	2.41
Peru	01/06/1980-01/12/2002	10,452	17,579	8,452	433	8	2.40
Indonesia	18/03/1996-31/12/2001	376	1,393	998	89	5	2.50
N. Ireland	14/07/1969-31/12/2001	2,698	3,523	3,523	2,698	1	3.17

Table 2.

What if we could then extend the statistical approach to predict the number of deaths to occur during such a conflict? Better yet, what if we could predict the number of deaths to occur due to a specific military operation? Would we try to avoid the casualties? Would our attempts at evading death be successful? Or would our efforts be in vain?

At the same time, what if these casualties were necessary for our overall success in a war? Would we then continue our military operations, knowing the amount of people who would die? Would we be consciously sacrificing people and view them as mere numbers in the war?

Conversely, what if the enemy had the same predictive power? What if they knew just how many resources were necessary to increase their chances of killing a certain amount of people? What if they knew how to statistically maximize that likelihood? What if they knew how frequently they needed to engage in attacks that resulted in a certain casualty size in order to augment their destructive power?

Math is powerful; there's no question about it. Knowing how and where to apply math is key to unlocking an unfathomable expanse of knowledge. At the same time, we must be careful. With knowledge comes responsibility, for we must stay true to our moral compasses. After all, who knew that math could predict death?

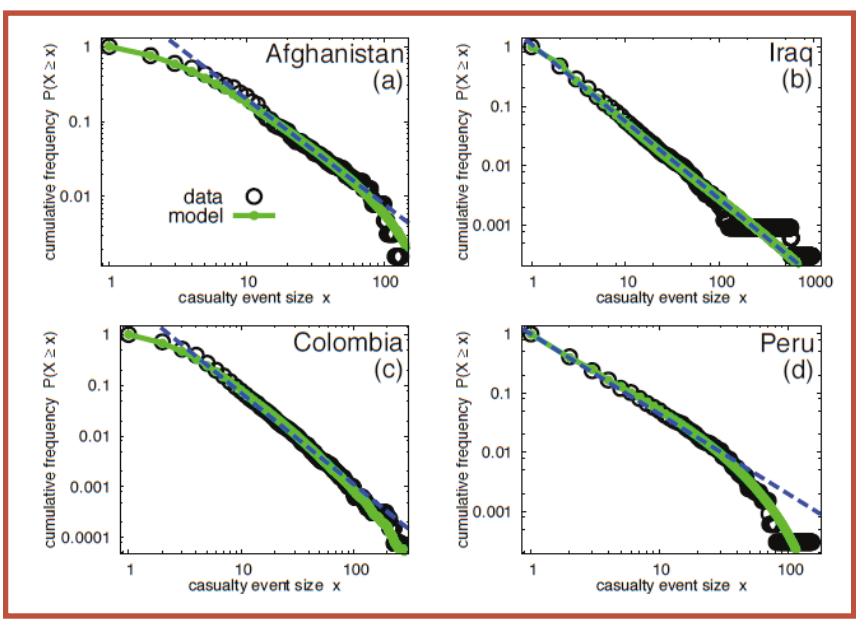


Fig. 2. Size of events

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MATHEMATICAL ASPECTS OF ELECTORAL SYSTEMS

CLAUDE TARDIF, Royal Military College of Canada

THERE IS A GROWING AWARENESS among Canadians that electoral systems used in other democracies can be very different from our own and the question arises as to whether ours, inherited from the nineteenth century, should be revised. Through four referendums on electoral reform at the provincial level, it has become obvious that the general public has little understanding of the purposes and workings of electoral systems in general, including our own. So who should educate? Politicians who win with the current system? Advocacy groups that would benefit from a change? Political scientists? In this brief sample of the subject I hope to show that electoral systems have a mathematical flavour and could be part of the high school mathematical curriculum.

Location problems and electoral methods

Figure 1 shows the population of seven towns and the road network connecting them. A hospital will be built in one of the towns, serving all of them. Where should it go? The *mode,* with the largest population, is G, the *median,* which minimizes the average travel distance, is *B* and the *center,* which minimizes the maximum travel distance, is *D*. Which criteria are more relevant? We will not answer this question directly, but rather show that the problem of choosing an appropriate electoral system has the same flavour. We will first see what happens if we let the population choose the location of the hospital democratically.

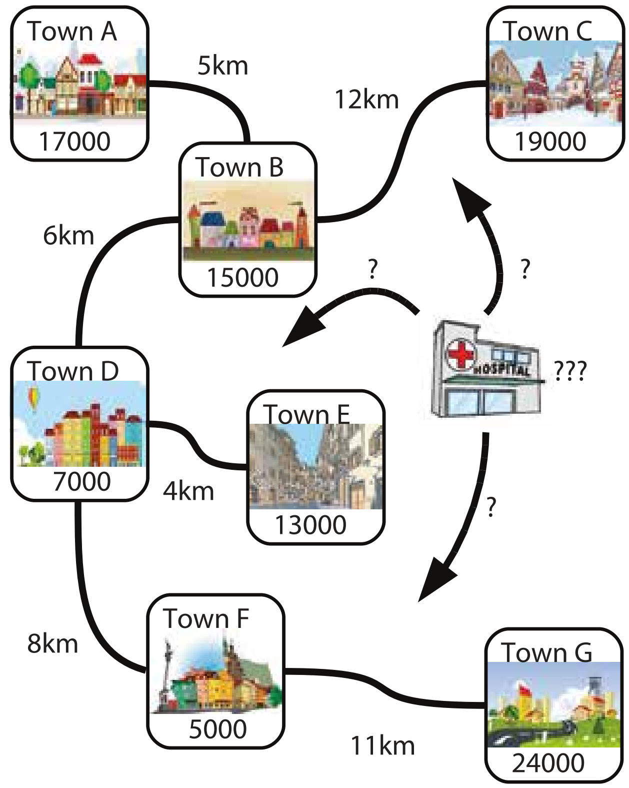


Fig. 1. Where should the hospital go?

G is the plurality winner: If everybody votes sincerely (each for their own town) and the winner is the town with the most votes, then *G* wins. This is the plurality procedure, which is a distributed way to compute the mode.

C is the two-round winner: If there is a second round between the two top place-getters, then *C* and *G* will compete in this second round. If everyone then votes for the town closest to them, then *C* will beat *G* by a score of 71000 to 29000.

A is the alternative vote winner: The multi-round process can be carried out to the extreme, with each round eliminating only the lowest place-getter: First *F*, then *D*… At each new round, each voter votes for the closest town still in the running, until one town gets fifty percent of the vote. In this case *A* would win after five rounds, as shown in the table below. Note that it is not necessary to hold the successive rounds at different times. The voters can fill a single ballot with their preferences listed in order. For instance, a resident of town *D* would list the towns in the order: *D*, *E*, *B*, *F*, *A*, *C*, *G*. The counting can then be performed iteratively by transferring votes from the candidates that are eliminated.

ROUND	1	2	3	4	5
A	17%	17%	17%	32%	51%
B	15%	15%	15%	–	–
C	19%	19%	19%	19%	–
D	7%	12%	–	–	–
E	13%	13%	20%	20%	20%
F	5%	–	–	–	–
G	24%	24%	29%	29%	29%

This electoral procedure is called *instant runoff voting* in the United States and the *alternative vote* in other parts of the world. Ballots where the candidates are ranked in order of preference are called *ranked ballots*.

These three *majoritarian* systems give three different winners and allow us to gradually introduce the ranked ballot, which is implicit in such location problems. None of these systems elects a central "consensual" location like the median or the center. Other systems fill that purpose:

B is the Condorcet winner: When the road network is a tree, as in this example, the median coincides with the *Condorcet winner*, that is, the candidate that wins against any opponent in a two-member race (when such a candidate exists). Indeed, a road joining neighbouring towns X and Y corresponds to a partition of the population into those that live closer to X than to Y and those that live closer to Y than to X. If more than 50% of the population lives closer to X, then X would win a majority against any town on the Y side of the network, and also moving the hospital from Y to X would reduce the average travel distance. If we represent this by an arrow from Y to X we get the diagram of Figure 2, showing that B is the median and the Condorcet winner.

D is the Borda winner: In the Borda system, a voter's ranked ballot is used to give points to the n candidates; the k-th favorite candidate gets $n - k$ points. The scores are counted and the candidate with the most points wins. The towns that are "somewhere in the middle" get points from all around, much more than the towns in dead ends. The winner ends up being D. However the center and the Borda winner are not guaranteed to always be the same, even in a tree network.

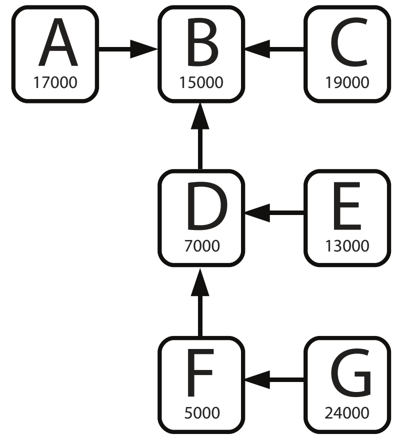


Fig. 2. Condorcet winner

The five electoral systems we have seen each give their own answer to what is an acceptable compromise position for a shared resource, sometimes coinciding with mathematically defined centrality measures. In real life the criteria for the best hospital location are more complex and towns are not dots along a tree. Nonetheless, our example describes fairly accurately the final rounds of the 2009 mayoral election in Burlington, Vermont, as illustrated in Figure 3. The method used was the alternative vote (called there, *instant runoff*). The "central" Democrat with the fewest votes was closer to the Progressive "left" than to the Republican "right", so the Progressive ended up winning the election. The Democrat was the Condorcet winner and the Republican was the plurality winner. This is a real-life example of three different "democratic" systems giving three different winners.

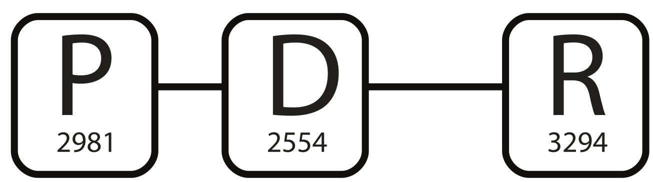


Fig. 3. 2009 Burlington mayoral election

So which system gives the best results? The answer may be obvious to unconditional supporters of the Progressives, the Democrats or the Republicans, but if we put aside partisan interests the question begins to look like the hospital location problem (with the road map more fuzzy and perhaps different from voter to voter).

It is interesting to note that in 2010, the Burlington instant-runoff voting was repealed in a referendum. The repeal campaign used the slogan "Keep voting simple;" implying that anything other than simple plurality is too complex. So it appears that the basic logistics of various electoral systems and the role they fulfill needs to be explained, much more than the deeper mathematical aspects of the subject.

Social choice theory and its model

An electoral system is a mathematical object. It is a function, of the form

$$f : \mathcal{L}(A)^P \to A$$

Here, A is a set of *alternatives* or candidates. $\mathcal{L}(A)$ is the set of *linear orderings of* A, that is, the set of possible ranked ballots listing the members of A in order of preference. P is the voting population. The contents B of the ballot box after the election is an element of $\mathcal{L}(A)^P$. This is the data that f must use to determine the outcome $f(B)$ of the election. Social choice theory studies all possible functions of this type. The plurality rule, two-round system, alternative vote and Borda count of the previous section are possible choices for f, provided that a tie-breaking rule is available if necessary. The Condorcet method needs additional back-up rules in the case when there is no Condorcet winner. These are just a few options among countless possibilities. However, to limit the choice to sensible options, we can formalise "fairness" in terms of mathematical axioms that f should satisfy. The choices then dwindle rapidly. In particular, the well-known impossibility theorem of Arrow states that when there are at least two voters and three alternatives, no function $f : \mathcal{L}(A)^P \to A$ satisfies the three basic fairness axioms of *non-dictatorship*, *Pareto efficiency* and *independence of irrelevant alternatives* (see for instance (3)).

However remarkable Arrow's theorem is, it is unfortunate that it is often the only thing known to mathematicians about voting systems. Closing the subject of voting systems with the statement "no system is perfect" is like rejecting calculators because they cannot represent irrational numbers with perfect accuracy. For one thing, approximations of fair systems may still be worthy of scrutiny. For another, Arrow's theorem applies only to a model, and many real-life voting situations escape it.

For instance, along the lines of Arrow's theorem, we find May's theorem which states that when there are only two alternatives, the majority rule is the only "fair" voting system. Yet, consider the three friends of Figure 4 deciding on whether to have pizza or sushi for lunch. Despite May's theorem, it is likely that they will go for sushi. The pizza voters only have a slight preference for pizza, while the sushi voter has a big preference for sushi. This information is not registered by the ranked ballot, which only records ordinal preferences. This is outside of the model of May's theorem.

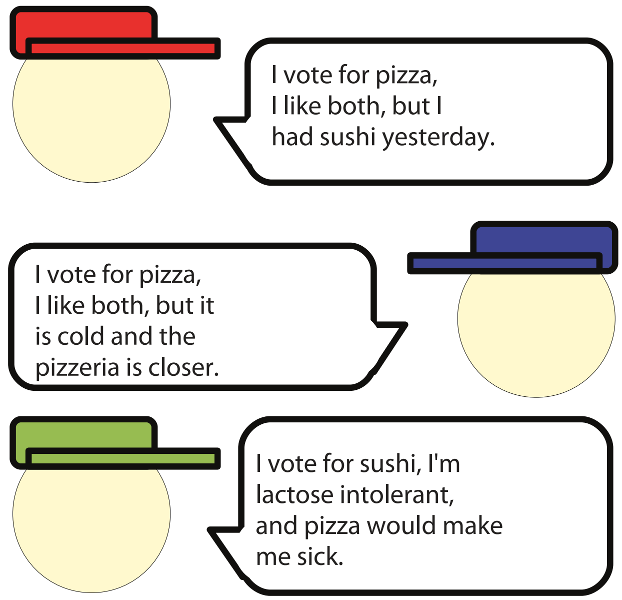


Fig. 4. Pizza or Sushi?

More significantly, often the outcome of an election is not an element of the set A of alternatives, but rather an element of the geometric realisation ΔA of the simplex with the elements of A as vertices. By this we do not mean a Frankenstein-type politician made from the limbs of the candidates, but rather an assembly of representatives. The axioms in Arrow's theorem do not apply to functions of the type

$$f : \mathcal{L}(A)^P \to \Delta A$$

which model our federal, provincial and city council elections. The only relevant "axiom" is the nonmathematical statement of Ernest Naville: "In a democratic government the right of decision belongs to the majority, but the right of representation belongs to all."

Representation in multimulti-member elections

Let's go back to the hospital location problem and see what happens if we let the population vote on the location of two hospitals instead of just one. Even with a *categorical ballot* where the voters can only put an X besides one or two candidates, we get more than one system:

G and C are the single non-transferable vote winners: If everybody votes for their own town and the two towns with the most vote win, then *G* and *C* win with 24000 and 19000 votes respectively.

***A* and *B* are the block voting winners:** If everybody gets two votes — one for their own town and one for the closest town — and the two towns with the most votes win, then *B* and *C* win with 51000 and 32000 votes respectively.

***B* and *F* are the sequential weighted block voting winners:** The *sequential weighted block voting* method uses the ballots of the block voting method, but the results are counted differently. First, the town with the most votes is elected, namely *B,* with 51000 votes. Then, the weight of every ballot with *B* on it is reduced by half. Thus the count of ballots for *C* drops from 32000 to 16000 and *F* wins the second hospital with 29000 votes.

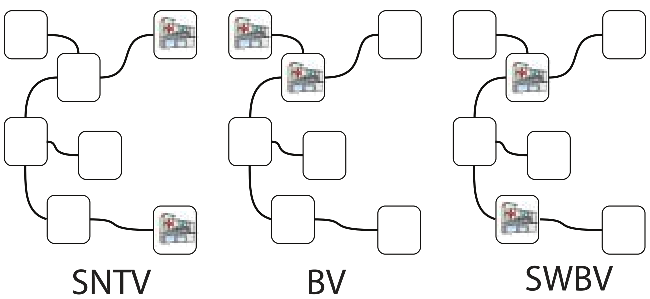


Fig. 5. Locations for two hospitals

The scores of the winning locations are higher with block voting than with other methods, but in Figure 5 we see that these locations are clustered in a narrow area rather than spread across the network. Indeed, when used in a political election, block voting usually produces a landslide for one party or a cluster of like-minded candidates and no representation for the others. Block voting is used for council elections in some Canadian cities, where it is called the *at large* system. It is instructive to see its outcome represented in a location problem as in the present example.

The *single non-transferable vote* is used in some elections around the world. However in real life, nominations would be tactical, to avoid vote splitting. The sequential weighed block voting is a theoretical method, not used in real elections. It illustrates the fact that the results are spread more evenly when the role of the ballots are equalised as much as possible. With sequential weighed block voting, 80% of the ballots count toward the election of a location, while with block voting, only 51% of the ballots count towards the election of a location and among them, 32% count towards the election of the two locations. It would be interesting to compile similar statistics with the results of at large elections in Canadian cities. Unfortunately the raw data that would be needed is never published.

The *single transferable vote* is a commonly used system that equalises the role of ballots to some degree. It uses a ranked ballot and works similarly to selecting the last survivors in the alternative vote. In our example, the latter method would give *A* and *G* as the winning locations. Indeed, after five rounds the surviving towns are *A* with 51000 votes, *E* with 20000 votes and *G* with 29000 votes. In the next round *E* is eliminated, so that the last two survivors are *A* and *G*. Note that the final tally is 71000 votes for *A* and 29000 votes for *G*, which means that 71% of the population is closer to the hospital at *A* and 29% is closer to the one at *G*. Therefore, it would make sense to build a much larger hospital in *A* than in *G*. However, if the hospitals must have equal size and each serve roughly half of the population, then the hospital at *A* will be overcrowded and the residents of *A*, *B*, *C*, *D* and *E* will often have to go to the other hospital. It would then make sense to let them have a say on its location, which the alternative vote does not allow.

The *single transferable vote* solves this problem as follows: A threshold of votes needed to be elected is established, namely $\lfloor v/(n+1) \rfloor + 1$, where v is the total number of votes and n is the number of candidates to be elected. In our example, this threshold is 100000/3 = 33334. An elected candidate only keeps this threshold number and the excess is transferred. After the fifth round, when *A* is elected with 51000 votes, it keeps only 33334 of these votes and the remaining 17666 votes are transferred to the next surviving preference, namely *E*. The count is now 33334 votes for *A* (elected), 37666 votes for *E* and 29000 votes for *F*, so *E* is the second location elected.

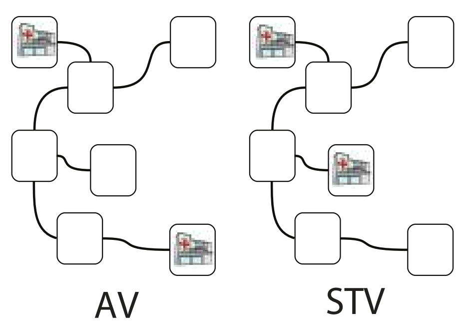


Fig. 6: More locations for two hospitals

The use of the single transferable vote is widely spread around the world. Ireland uses it for all its elections and calls it *proportional representation*. Indeed, when used in an election with a party system the results are fairly proportional, in the sense that the proportion of members a party elects is roughly equal to the proportion of first place votes that the party gets. Elsewhere in the world the single transferable vote also tends to produce fairer results in terms of gender and race representation: Australia uses the alternative vote for its *lower house* and the single transferable vote for its *upper house*. The proportion of women elected is consistently higher in the upper house. In the middle of the 20th century many American cities used the single transferable vote for municipal elections allowing some minorities to gain representation for the first time. In particular, this is how New York City got its first African American councillor, Adam Clayton Powell, in 1941. Unfortunately, the representation of minorities was met with growing intolerance and there were repeated repeal campaigns. In Cincinnati, the fear of some day having a black mayor was the theme of the successful repeal campaign in 1957 (see (1)).

In British Columbia, Canada, the single transferable vote was proposed as an alternative to the current single member plurality system for provincial elections. It was rejected in two referendums — in 2005 and 2009. Rather than the effect on the representation of minorities, it was the perceived complexity of the system that worked against it. In fact, the system is indeed more complex than our example suggests: when ballots are transferred away from an alternative that has already reached threshold, the question arises as to exactly which ballots should be transferred. In our example, when ballots are transferred away from *A*, the next preference expressed is always *E*, so that the choice of ballots transferred is irrelevant. However it is possible to make up with examples where the choice of ballots transferred can modify the outcome.

The traditional way to solve this dilemma is to randomly choose the ballots to be transferred — Ireland still uses this method. When the voting population is large enough the chances of having the outcome affected by a particular random draw is negligible. However, the method proposed in British Columbia was instead to transfer a fraction of every ballot, relying on modern technology to ease the calculations. This made the system harder to explain to the general population and the complexity of the system overshadowed its purpose and potential benefits.

With a party system in place, fair representation can be achieved with systems that are more simple to explain than the single transferable vote.

Representation and party systems

Canada and all of its provinces use *single member plurality* to elect their legislatures. This amounts to holding parallel elections in each electoral district using the plurality method. It is well known that this method distorts the results, in the sense that the proportion of members a party elects differs significantly from the proportion of the popular vote it gets. Typically, a party wins a majority government with much less than 50% of the popular vote. In extreme cases, the party that makes up the government is not even the party that obtains the largest fraction of the popular vote. This phenomenon is easily illustrated by the following example.

U	U	U
U	N	N
U	N	N

In the above table there is a majority of *U*s, but if we consider the rows of the table as the ridings in a single member plurality election, we get a majority of rows with a majority of *N*s, resulting in a “majority” of *N*s. Such “wrong winner” elections happened a dozen times in Canadian provincial elections since World War II. However the letters *U* and *N* in our example are taken from the notorious election of 1948 in South Africa. The United Party won the largest fraction of the popular vote, but the National Party elected more members and formed the goverment. It had been campaigning on its newly devised program of apartheid. As a result, systematic segregation took hold of the country for half a century.

We will consider the systems where electoral districts are merged as multi-member districts. The *district magnitude*, or the number of members per district, is an important design specification of such systems. The districts can be paired as two-member districts as in Chile or the whole country can be a single district, as is done in Israel. However, these extremes are not common. According to the ACE Electoral Knowledge Network *(http://aceproject.org)*, most scholars agree that district magnitudes between three and seven members tend to work quite well.

For instance, let's consider the results in the five districts of Quebec City in the 2011 federal election.

	B	C	L	N	O
D1	10250	13845	3162	24306	1196
D2	8732	16220	3505	24131	1021
D3	14640	13207	8110	23373	1139
D4	8148	21334	3612	22629	1032
D5	14684	9330	4735	22393	1372

The major parties contesting the election were *B* (Bloc), *C* (Conservative), *L* (Liberal) and *N* (New Democratic Party). (*O* stands for other.) Since single member plurality was used, the *N* party, which had a plurality of the votes in each of the five districts, elected five representatives with 42% of the popular vote. Parties *B*, *C* and *L*, with respectively 20%, 27% and 8% of the vote, elected none.

If the five districts were merged as one, each party could be given a proportion of elected candidates roughly equal to the proportion of the popular vote it receives. If there are n candidates to be elected, then each party would elect one candidate for each slice of $(100/n)$% of the popular vote it receives. In the case of Quebec city, this yields one candidate elected for each 20% of the popular vote: one for *B*, one for *C*, none for *L* and two for *N*.

There are different methods of rounding to elect the fifth candidate. Let p and c, respectively, be the percentage of popular vote a party received and the number of candidates it has already elected. According to the *Hare method*, the last seat is given to the party with the largest remainder $p-c \cdot 20\%$, namely *L* with a remainder of 8%. According to the *D'Hondt method*, the extra seat is given to the party with the largest quota $p/(c+1)$, namely *N* with a quota of 14%. These two systems of proportional representation respectly give results of 2*N*, 1*B*, 1*C*, 1*L* and 3*N*, 1*B*, 1*C* instead of the result of 5*N* obtained with single member plurality.

Which of these is the fairest outcome? There are no mathematical axioms modeling fairness for multi-member elections. There are however, quantitative measures of the outcome, such as the Gallagher's *least square index* which measures how proportional the result of an election is (see (2)). There is also a (fractional!) measure of the "effective number of parties" (see (4)); if the election is fair, this number should be roughly the same before and after the election. However, it is better to use such measures on the global result of the election rather than on a single five-member riding.

In our example, the Hare method gives party *L* one member, that is 20% of the representation, with only 8% of the popular vote. This is a lucky break that depends on the distribution of the popular vote among the other parties. Across many districts things tend to even out and the results can be fairly proportional.

An interesting way to illustrate this point is to note that our example is amenable to the *Alabama paradox*. Suppose that Quebec City had been a six-member riding instead of a of five-member riding. Then, the share of votes needed to elect one candidate would have dropped to $16\frac{2}{3}$%. Hence *N* would have gotten two candidates elected and *B* and *C* would each have one. With the Hare method, the two remaining seats would have been given to the two parties with the largest remainders, namely *N* and *C*. Thus, party *L* would have lost its representative even though the number of representatives increased!

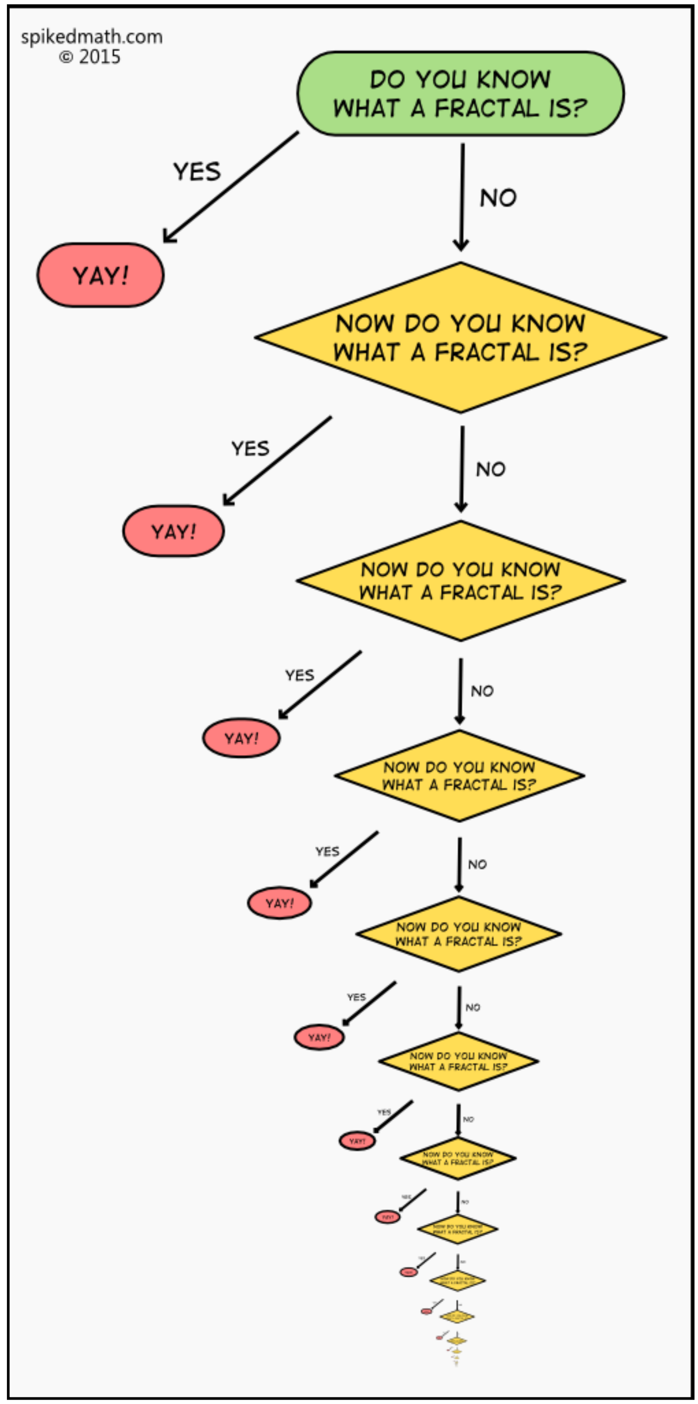
The Alabama paradox scenario cannot happen with the D'Hondt method; increasing the number of representatives could never decrease a party's representation. However with the D'Hondt method, parties with a lower level of support tend to be underrepresented, unless the district magnitude is much larger.

There remains the question of which specific candidates of each party should be elected. There are many options. In *closed-list systems* the parties decide the order in which their candidates are elected. Such systems are prefered by special interest advocacy groups because the lists can be used for positive discrimination. On the other hand, *open list systems* allow voters to have a more active role in deciding which candidates get elected. There are *mixed systems,* in which some candidates are elected from single-member districts and others from a list. This allows independent candidates to run with a fair chance of being elected. The single transferable vote of the previous section gives independents a chance and allows a voter's preferences to run across party lines.

Conclusion

We often hear the comment that the math taught in school should be more relevant to real life. The study of electoral systems connects to the politics that fills the news. Talking about electoral systems to the general public is different from teaching math in a classroom; it can confront people with their poor understanding of ratios and proportions and with the fact that others may understand the system better than themselves. People can get defensive and it can be difficult to keep the focus on electoral systems rather than political gossip.

As we have seen, the presentation of electoral systems can use combinatorics and algorithms to explain different systems and their purpose, algebra and topology for general models, ratios, proportions and statistics for analyzing the results. It can be taught at an abstract or a pragmatic level. It can expose a degree of systemic discrimination inherent to a system, which would be considered outrageous if it were on a sign in a public space or on a job ad rather than hidden in the intricacies of an electoral machinery. Perhaps these aspects of the topic are better suited to the mathematics curriculum than to the civics or politics curriculum.



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AN EASY WAY TO SPEED UP THE ERADICATION OF POLIO

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IN THE ENTIRETY OF HUMAN HISTORY we've only eradicated two diseases: smallpox (which used to kill a great many children) and rinderpest (a cow disease). However, several more diseases are in line for eradication in the next few years: guinea worm disease (a non-fatal disease spread by drinking water) and polio. We have a vaccine against polio.

SO WHY ISN'T POLIO ELIMINATED YET?

Polio used to be widespread; the longest-serving President of the United States, Franklin Roosevelt, suffered from it and was mostly confined to a wheelchair (although he didn't want anyone to know, so often attended public events while seated in a car). When eradication efforts began in 1988, there were 350,000 cases. Now there are fewer than 500. Polio can cause severe disability and paralysis, especially among children, although there are a large number of cases without symptoms. The disease is spread via direct contact with infected faeces, as well as through water reservoirs. The virus can't reproduce in water, so this form of transmission is slower.

This means that mathematical models must take into account both time and space. The disease changes in time, as the numbers fluctuate between seasons, but space is also important and in particular, how and where people are distributed. One way that mathematical modelling can deal with this is through what's called a metapopulation model. This involves breaking space into different regions or patches and developing a system of differential equations (our model) within each one. We then link each patch together according to the population movement. By thinking of each region (whether national, provincial or smaller) as a "patch," it is possible to model the dynamics both within and between different patches. One of the key dynamics we're interested in is vaccination.

When invented in the 1950s, the polio vaccine was hailed as an enormous breakthrough. The first vaccines had to be injected, but a few years later there was another breakthrough: an oral vaccine. **Why is this so important?** Because it can be administered easily and painlessly to children, without requiring a doctor. This makes mass vaccination possible.

The oral vaccine was responsible for the large decline in incidence over the past 30 years – in 2009, there were 1606 cases and by 2012, there were just 223. However, something happened to cause that number to double in 2013.

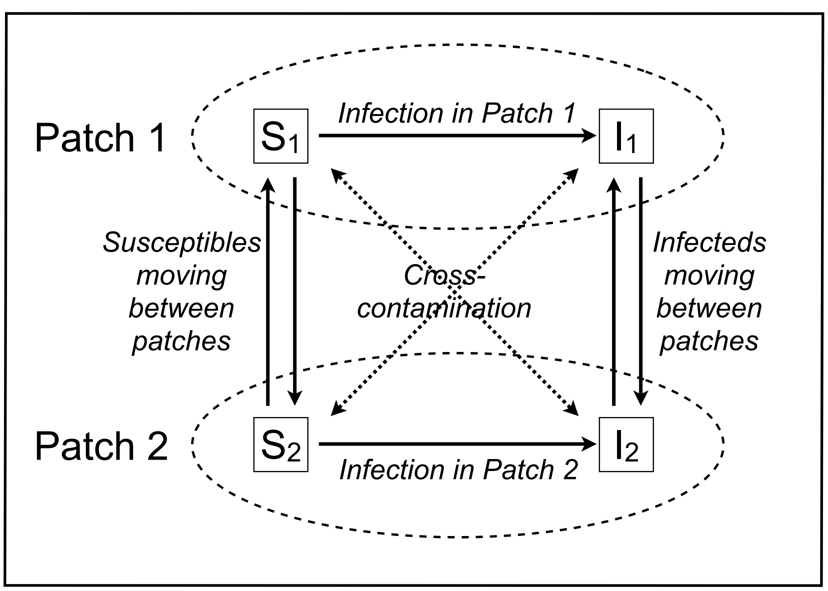


Fig. 1. Susceptible and infected individuals in two patches, with cross-contamination.

WHERE DID THESE CASES COME FROM?

The problem with vaccines is that no one likes taking them. We would like other people to be vaccinated, but not ourselves. When a disease starts to disappear people stop getting vaccinated, because they aren't seeing its effects. Worse, ignorance of the benefits of vaccination have led directly to increased outbreaks. In 2003, Nigeria issued a fatwa stating that vaccines were an American plot to sterilise Muslims. (They aren't.) This caused the number of Nigerian cases to increase.

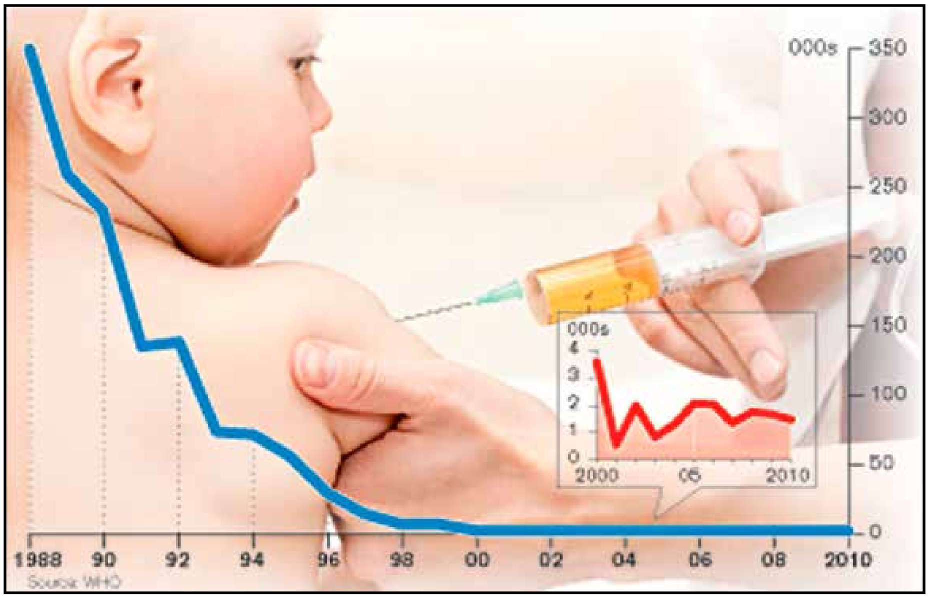


Fig. 2a. Estimated number of polio cases per year

As well, aid workers administering the vaccine in Pakistan and Afghanistan have been murdered, thus decreasing the number of vaccinations administered and causing the incidence to rise.

Another challenge to eradication is the vaccine itself. In a very small number of cases (1 in 750,000), the vaccine can give you polio. However, we have been so successful at controlling wild polio (i.e., the polio strain that spreads in nature, as opposed to vaccine-induced polio) that 15% of all 2013 cases were due to the vaccine.

At the same time, there have been great advances – India was declared polio-free in March 2014 and the disease now exists in only a handful of countries. There has been a 99% drop in the number of cases since eradication efforts began.

We're close to eradicating this disease.

HOW CAN WE IMPROVE OUR EFFORTS?

One of the best ways that we've found to vaccinate individuals against diseases such as polio and measles is through National Immunisation Days (NIDs). These involve mass vaccinations in a one- to two-day period. To give you an idea of how large these are, in a single NID in India, 174 million children are vaccinated, using 225 million doses, employing 2.5 million vaccinators. These happen twice a year and are called "pulses." Most countries with polio have these NIDs. They are a way of ensuring that most everyone gets vaccinated, because it can be done in schools or villages all at once.

However, in most countries health is managed locally (such as at the provincial level, as happens in Canada). Even when countries manage health nationally, they don't coordinate with neighbouring countries.

SHOULD THEY? MORE FORMALLY, SHOULD WE SYNCHRONISE PULSE VACCINATIONS?

There are good reasons to think this might be important. Many people migrate across borders, carrying diseases with them. Different regions have different transmission seasons, depending on climate, geography etc and environmental transmission may not respect borders (such as a lake that joins two countries).

Suppose you're a migrant worker who moves between two areas: Patch 1 and Patch 2. If you are in Patch 1 when Patch 2 is being vaccinated, but have returned to Patch 2 when they're vaccinating in Patch 1, you will not receive the vaccine.

HOW CAN WE MAKE SURE ALL MIGRANTS ARE VACCINATED?

By vaccinating all the patches at once, of course!

More formally, mathematical modelling shows that there is a local minimum ($x'=0$ and $x''>0$) when the phase difference between two patches is zero.

Fig. 2b. The exponential decline of polio is largely due to the oral vaccine.

That is, the disease spread is least when vaccinations in different patches occur at the same time. In fact, this can be the difference between eradicating the disease and having it persist. (See Figure 3.)

We measure the progress of eradication using R_e, the effective reproduction number. This is a measure of disease spread, taking into account the average number of secondary infections. Mathematically, we derive this number by measuring the stability of an uninfected equilibrium. That is, if there is no disease, then there will continue to be no disease (it can't spontaneously appear) unless someone brings it in from outside. This is equivalent to perturbing the equilibrium, just slightly. If those perturbations result in the disease returning to its uninfected state, then we say that the equilibrium is stable. If not (i.e., if a few cases can lead to an outbreak), then the equilibrium is unstable. Mathematics is so useful here because it can predict this number in advance, such that we can tell whether our intervention methods are likely to be successful. If they stabilise an otherwise unstable equilibrium, then there is a good chance of eradicating the disease.

WHAT HAPPENS IF WE ADD OTHER FACTORS?

Migration makes the spread of disease more likely, making it even more important to synchronise the vaccinations; the more migration there is, the more crucial this becomes. (See Figure 4a.) We want to vaccinate in the blue areas, but as migration pushes the curve to the back, the only blue areas are at the sides, when the patches are in phase.

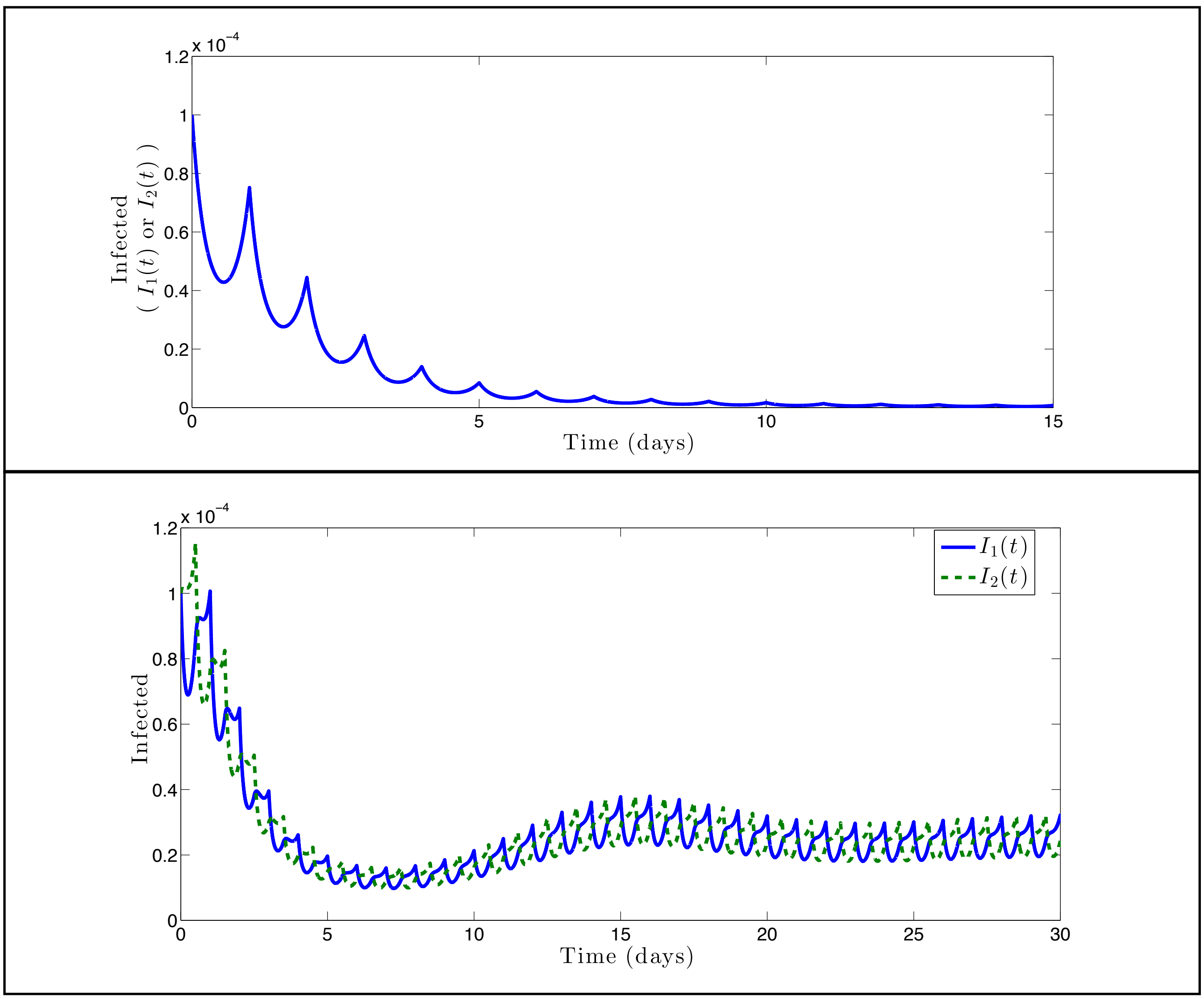


Fig. 3. These two figures use identical data, except that the vaccinations are out of phase. When they happen at the same time we have eradication – when they don't, the disease persists in both patches.

Seasonality adds a complication: we need to vaccinate when the transmission is low. (See Figure 4b.) Now we want to vaccinate in the blue corners. So we still want to synchronise the vaccinations, but we also want to vaccinate at the right time of year.

BUT WHAT IF TWO PATCHES HAVE DIFFERENT SEASONAL EFFECTS? THAT IS, WHAT IF SYNCHRONISING THE VACCINATIONS WOULD CONFLICT WITH THE LOW-TRANSMISSION SEASON IN ONE OF THE PATCHES?

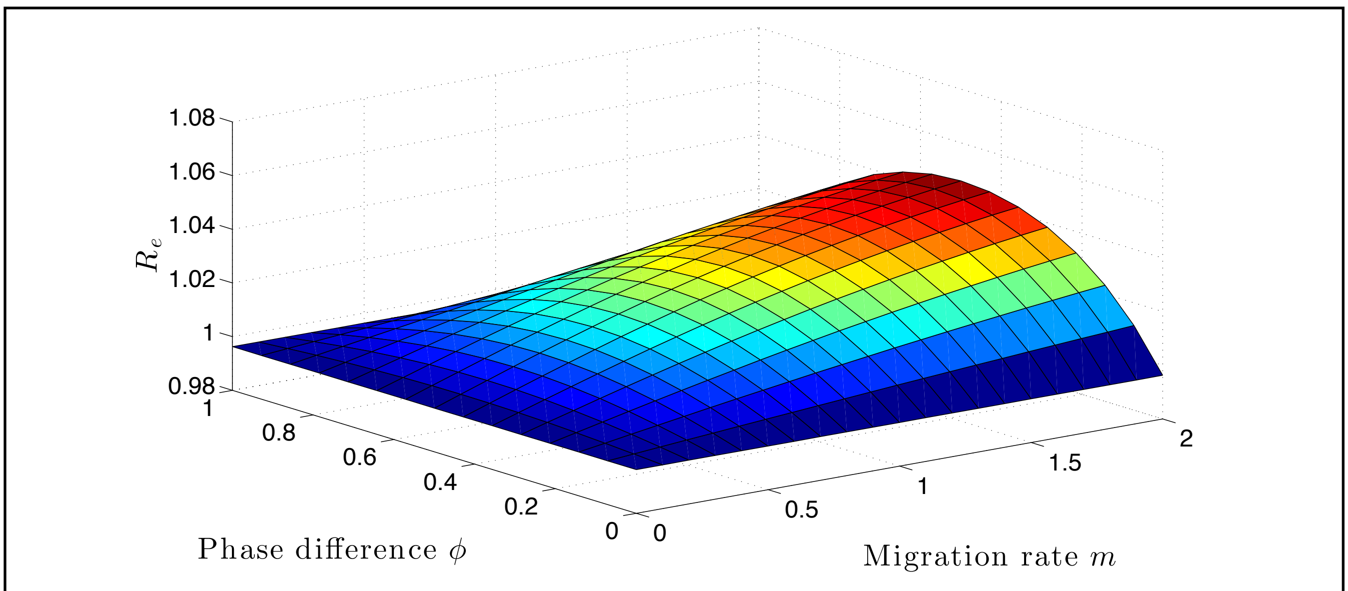
This issue has arisen in the past, with Operation MECECAR – a program to coordinate vaccinations in the Mediterranean (ME), Caucasus (CA) and Central Asian Republics (CAR). They considered this problem and decided to synchronise the vaccinations.

WAS THIS THE CORRECT STRATEGY?

No.

Mathematical modelling shows that the answer is more subtle and depends on migration. If migration is low, then the pulses don't need to be synchronised. In this case, the best strategy is to de-synchronise the pulses and vaccinate in the low-transmission season for each region (i.e., the blue valleys). (See Figure 5a.) If migration is high, the strategy is completely changed. In this case, migration overwhelms seasonality and the pulses need to be synchronised again. (See Figure 5b.)

Thus, modelling shows that the best strategy is to synchronise the pulses in almost all cases. The good news is that this is something we can do.



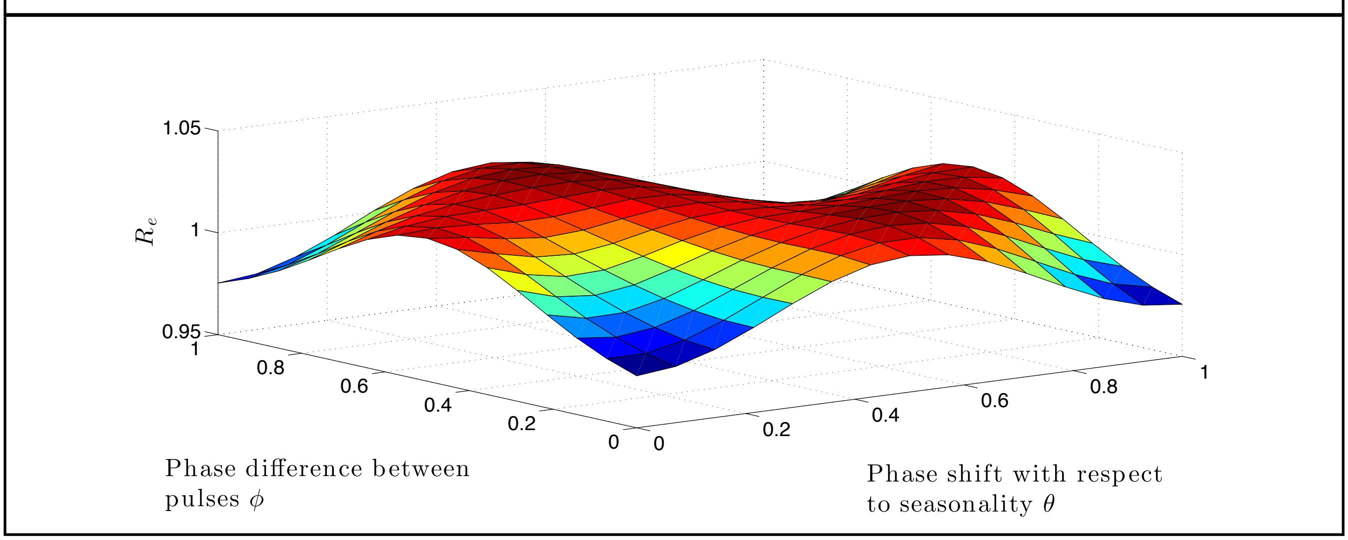


Fig. 4a. The more migration, the more important synchronisation becomes.
Fig. 4b. If the virus fluctuates seasonally, then we want to vaccinate in the blue corners; i.e., when the vaccines are synchronised and transmission is low.

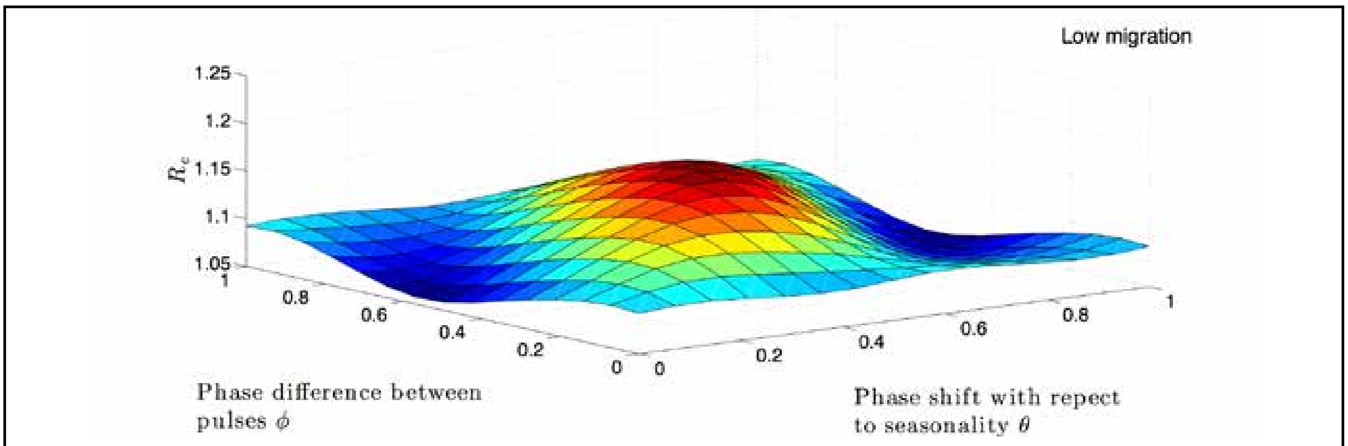
Coordinating NIDs across different regions is within our power and provides obvious benefits. Modelling also helps us see the most important factors that affect polio eradication: migration and seasonality. Individually these lead us to synchronise the pulses; however, if they overlap, then we must de-synchronise the pulses if migration is low, but synchronise the pulses if migration is high. These conclusions aren't necessarily obvious without modelling. As we saw from Operation MECACAR, making decisions without models can lead to the wrong strategy – and this affects people's lives.

Very soon, we stand a chance of doubling our success at disease eradication, bringing our total up to four. The final push to eradicate any disease is difficult, but it is important not to lose momentum – or hope. In the home stretch of polio eradication, synchronising vaccinations across regions may be the key to removing one of humanity's greatest scourges once and for all.

Figure 2a is from the World Health Organization

Figure 2b was taken by Rod Curtis for the World Health Organization

Figures 3–5 are from the original academic article: Cameron J. Browne, Robert J. Smith?, Lydia Bourouiba, 2015. From regional pulse vaccination to global disease eradication: insights from a mathematical model of Poliomyelitis. *Journal of Mathematical Biology,* **71**(1), 215–253.



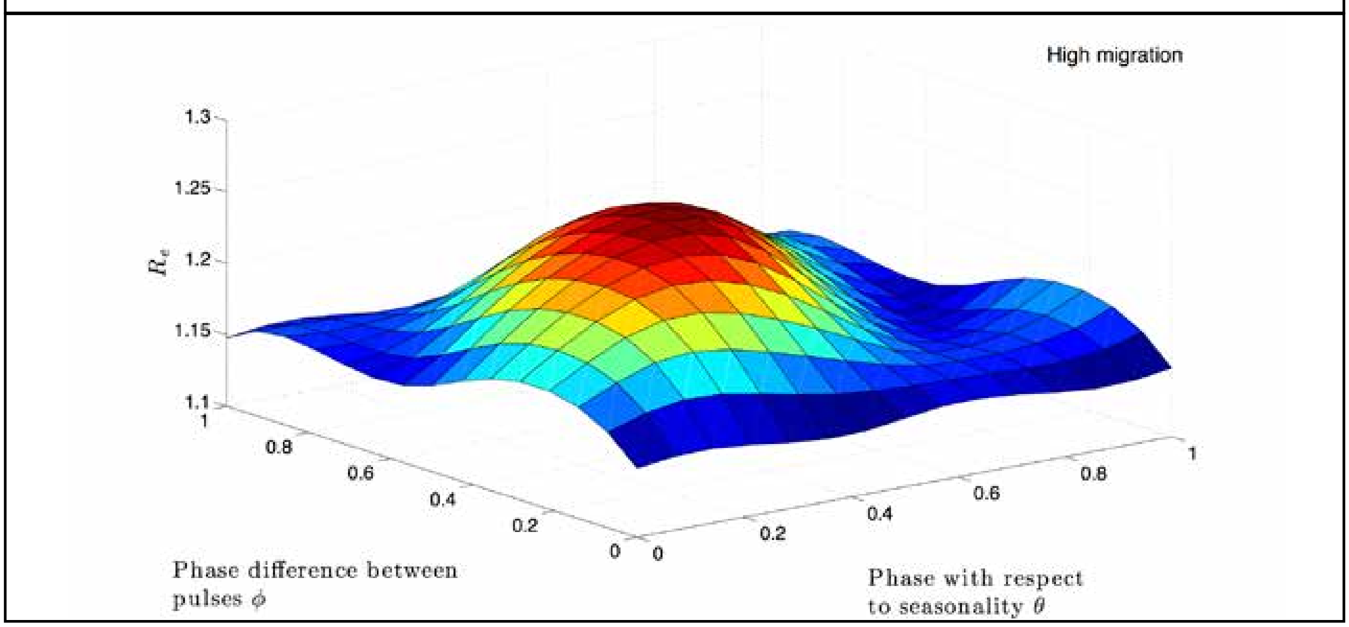


Fig. 5a: If migration is low, the pulses should be de-synchronised and vaccination should occur in the low-transmission seasons for each patch.
Fig. 5b: If migration is high, then this overwhelms seasonality and the pulses should be synchronised.

Problems

1. Find all positive integers *n* for which $\sqrt{5n+2}$ is an integer.

2. Find all the integers *m* such that $m^2 + 3m - 50$ is divisible by 19^2.

3. Find the number of positive integers ≤ 1000 which are not divisible by any of 5, 7, and 11.

4. Let *x*, *y* be any real numbers. Find the smallest possible value of
$$|x-3|+|x-y+2|+|100-y|.$$

5. Let $P(x)$ be a polynomial of degree four and let $a \geq 1$, $b \geq 1$ be distinct numbers such that $P(a) = P(1 - a)$, $P(b) = P(1 - b)$. Show that $P(x) = P(1 - x)$, for any real number *x*.

6. Let $a_1, a_2, \ldots,$ an be real nonnegative numbers such that $a_1 + a_2 + \ldots + a_n = k$. Find the maximum value of $a_1a_2 + a_2a_3 + \ldots a_{n-1}a_n$.

7. Let $M = \{1, 2, 3, \ldots, 2016\}$ and k a positive integer. Find the minimum value of *k* for which any subset of *M* with *k* elements contains at least two distinct numbers such that one of them is a multiple of the other.

8. The point *P* is inside a convex quadrilateral *ABCD* of area 168 such that $PA = 9$, $PB = PD = 12$ and $PC = 5$. Find the perimeter of the quadrilateral.

★ ★

PRIZE!

PIMS is sponsoring a prize of $100 to the first high school student (from within the PIMS operating region: Alberta; British Columbia; Manitoba; Saskatchewan; Oregon; Washington) who submits the largest number of correct answers before September 1, 2016. Please submit to **pims@math.uvic.ca**.

★ ★

2014 MATH CHALLENGES

1. Let $n \geq 1$ be a positive integer. Determine the greatest integer less than or equal to $(\sqrt{n}+\sqrt{n+1})^2$.

SOLUTION

Let $A=(\sqrt{n}+\sqrt{n+1})^2=2n+1+2\sqrt{n(n+1)}$. Since $2n<2\sqrt{n(n+1)}<2n+1$ for any positive integer n we have $4n + 1 < A < 4n + 2$ and hence the greatest integer less than or equal to A is $4n + 1$.

2. Let m, n, p be integers. Prove that 12 is a divisor of $m^2+n^2+p^2$ if and only if 12 is a divisor of $m^4 + n^4 + p^4$.

SOLUTION

For any integer m, $m^4 - m^2 = m^2(m^2 - 1) = (m - 1)m(m+1)m$. If m is even then m^2 is divisible by 4 and if m is odd then $(m - 1)(m + 1)$ is divisible by 4. Hence, in any case, $m^4 - m^2$ is divisible by 4. Also, $(m - 1)m(m + 1)$ is divisible by 3, as a product of three consecutive integers. Consequently, since 3 and 4 are relatively prime we conclude that 12 is a divisor of $m^4 - m^2$ and also of $(m^4 - m^2) + (n^4 - n^2) + (p^4 - p^2) = (m^4 + n^4 + p^4) - (m^2 + n^2 + p^2)$. Now, if 12 is a divisor of $m^2 + n^2 + p^2$ this is equivalent to say that 12 is a divisor of $(m^4 - m^2) + (n^4 - n^2) + (p^4 - p^2) + (m^2 + n^2 + p^2) = m^4 + n^4 + p^4$.

3. Let $n \geq 25$ be an integer. Find the remainder obtained when $n(n + 1)(n + 2)$ is divided by $n - 2$:

SOLUTION

We have

$$\begin{aligned} n(n+1)(n+2) &= [(n-2)+2][(n-2)+3][(n-2)+4] \\ &= (n-2)^3 + 9(n-2)^2 + 34(n-2) + 24 \end{aligned}$$

If $n > 26$, the remainder is 24. If $n = 26$ the remainder is 0 and if $n = 25$ the remainder is 1.

4. The numbers 1, 2, 3, . . . , 2014 are arranged in a circle in cyclic order. We paint the numbers 1, 5, 9, and every fourth number, round and round the circle. Some of the numbers may be painted more than once. Find the number of numbers which will never be painted.

SOLUTION

Since 2014 is even, even numbers are never painted. Since 2014 is not divisible by 4, all odd numbers are painted. Hence the number of numbers which will never be painted is $2014 \div 2 = 1007$.

5. Find all the pairs (a, b) of positive real numbers such that

$$\frac{\sqrt{a}}{a+4}+\frac{\sqrt{b}}{b+4} \geq \frac{1}{2}.$$

SOLUTION

Since $\frac{\sqrt{a}}{a+4} \leq \frac{1}{4}$, $\frac{\sqrt{b}}{b+4} \leq \frac{1}{4}$ and the equal sine holds only if $a = b = 4$, there is only one pair $(a, b) = (4, 4)$ which veries the given inequality.

6. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x+\frac{3}{2}) \leq 2x \leq f(x)+3$, for every $x \in \mathbb{R}$.

SOLUTION

From the inequality $2x \leq f(x) + 3$ we obtain that $f\left(x+\frac{3}{2}\right) \leq 2x \leq f(x)+3$, and hence $2x \leq f(x+\frac{3}{2})$ which combined with the given condition $f(x+\frac{3}{2}) \leq 2x$ gives $f(x+\frac{3}{2}) = 2x$. It follows that $f(x) = f((x-\frac{3}{2})+\frac{3}{2}) = 2(x-\frac{3}{2}) = 2x-3$.

7. The value of a diamond is proportional to the square of its weight. A diamond breaks in two pieces and their total value is now 32% lower than the original value. Find the ratio of the weight of the larger piece to the smaller piece.

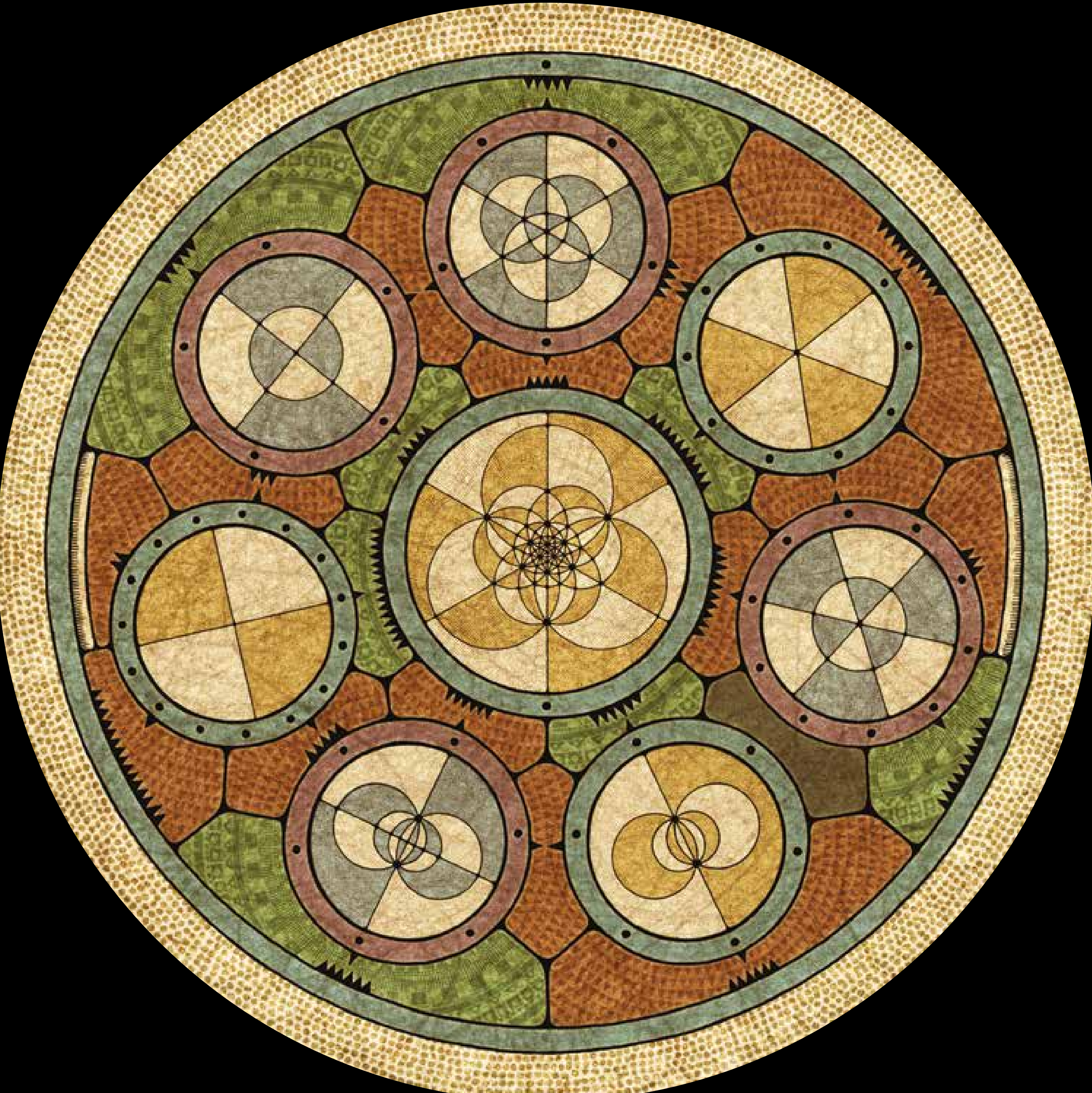
SOLUTION

Let the weight of the original diamond be 1, the weight of the larger piece be x and the weight of the smaller piece be y. Then $x + y = 1$ and $x^2 + y^2 = 0.68$. Eliminating y, we have $0 = x^2 - x + 0.16 = (x - 0.8)(x - 0.2)$. Since $x > y$, $x = 0.8$ and $y = 0.2$ so that $x : y = 4 : 1$.

8. Find the number of isosceles acute-angled triangles with perimeter 40 such that all three sides have integral lengths.

SOLUTION

Let m, m and n be the lengths of the sides of the triangle. Then $2m+n = 40$. Thus we may have $(m, n) = (19, 2)$, (18,4), (17,6), (16,8), (15,10), (14,12), (13,14), (12,16) and (11,18). However, (11,18) is not a convenient solution since $11^2 + 11^2 = 242 < 324 = 18^2$, so that the triangle is not acute. Hence, there are eight isosceles triangles with the requested properties.



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