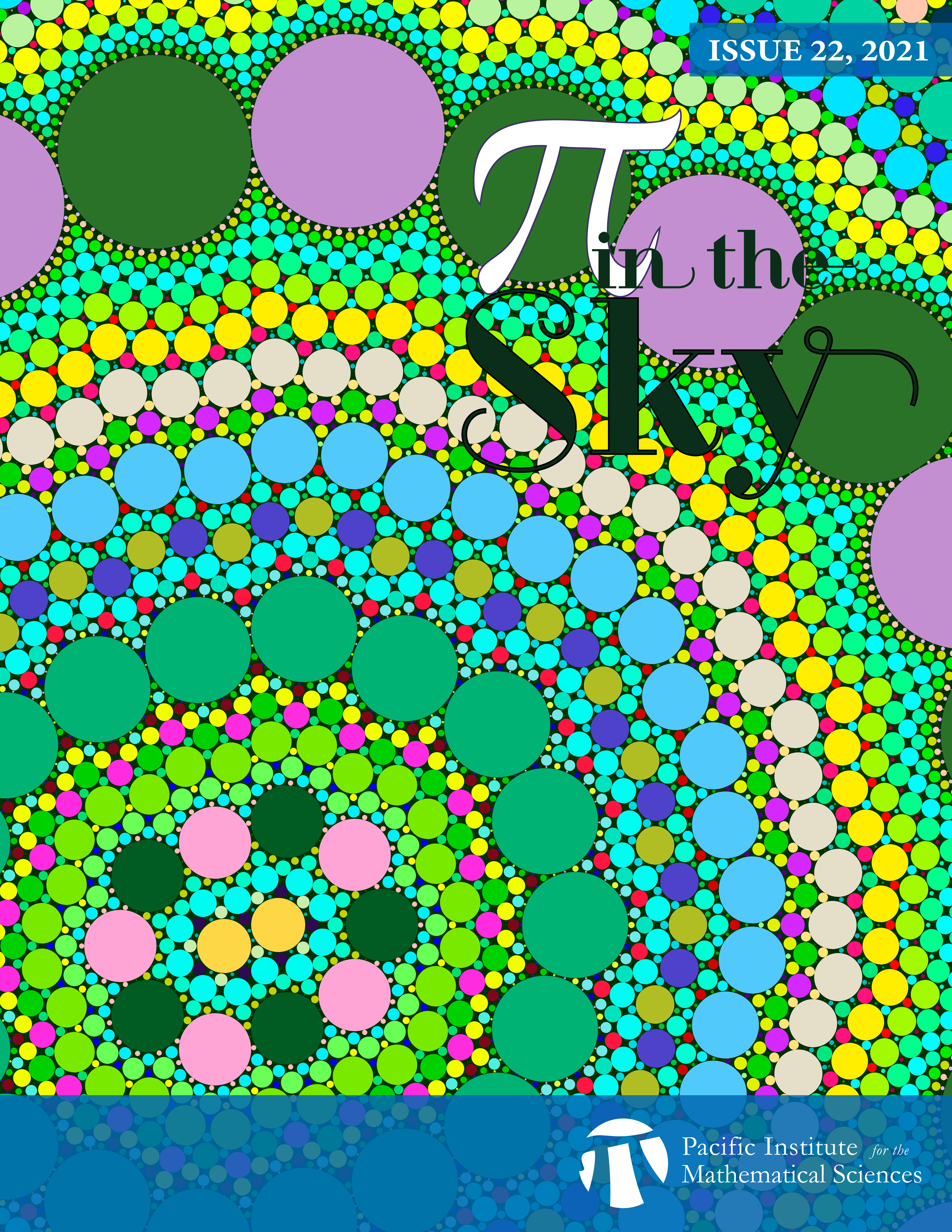
ISSUE 22, 2021

π in the Sky



Pacific Institute for the Mathematical Sciences

Pi in the Sky

Issue 22, 2021

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Pi in the Sky is aimed primarily at high school students and teachers, with the main goal of providing a cultural context/landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and articles may also put curriculum topics in a different perspective.

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A Note on the Cover

The image on the cover comes from a research project of mine with Yitwah Cheung and Arek Goetz. We were studying a "piecewise isometry". The idea is that you define a transformation of the plane where you shift the bottom half plane left by 1 unit; the top half plane right by 1 unit; and then rotate the whole plane by a fixed angle. When you apply this transformation repeatedly, all pairs of points stay at the same distance from each other (this is what the word "isometry" means) unless they land on opposite sides of the fault line (the x-axis). The discs in the picture consist of regions of the plane that never end up on opposite sides of the fault line. The research project consisted of understanding the patterns of discs that show up, and computing what proportion of the plane was covered by those discs.

There is also a movie of these patterns where the angle of rotation is gradually changed. That movie can be found at: https://www.math.uvic.ca/faculty/aquas/anim2.gif.

Anthony Quas
Managing Editor

Snarks

BY KIEKA MYNHARDT

Professor of Mathematics and Statistics, University of Victoria.

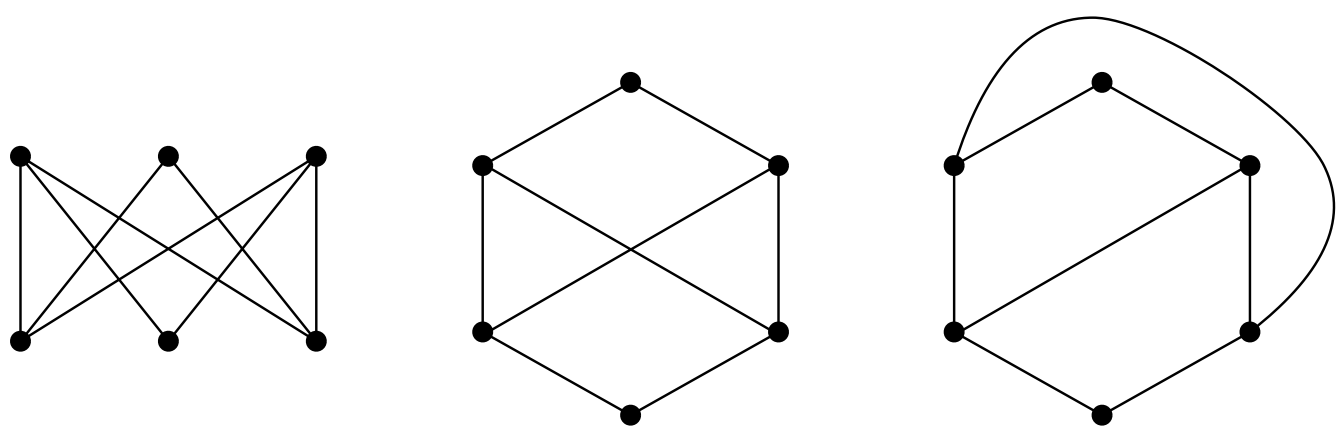
"They sought it with thimbles, they sought it with care;
They pursued it with forks and hope;
They threatened its life with a railway-share;
They charmed it with smiles and soap."

From: The Hunting of the Snark
by Lewis Carroll, circa 1876

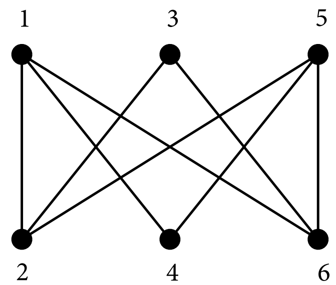
So, what is a snark?

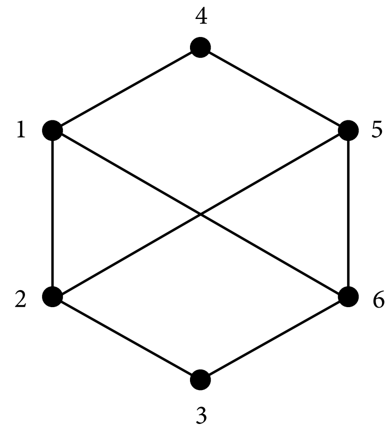
We don't know what Lewis Carroll meant it to be because nobody ever found one (although the Oxford English Dictionary says it's an imaginary animal). In mathematics, a snark is a very special kind of graph. So, what is a graph? Forget plots. For our purposes, a graph consists of a collection (let's keep it finite) of dots, called vertices (the singular is "vertex"), and a collection of lines, called edges, and each edge joins a vertex to another vertex. Google "What is graph theory?" to find a formal definition.

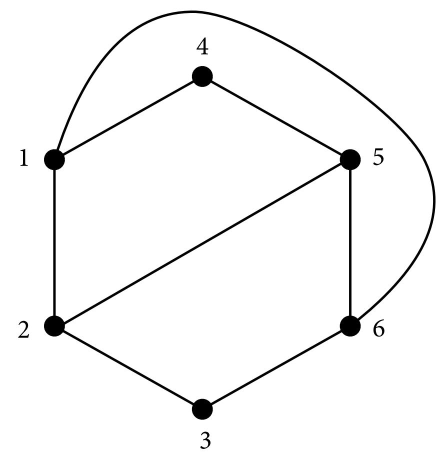
If the collection of vertices is reasonably small, we can draw the graph to show how the edges connect the vertices. The sketch below shows a graph drawn in three different ways. Note that edges need not be straight lines and may cross one another; there is no vertex at such a crossing point and edges meeting at a vertex don't cross there.



How do we know that the three graphs are the same? Geometry plays no role. The next sketch shows the same drawings, but now the vertices are labelled 1, 2; ..., 6. All we need to do is check that the edges are labelled with the same pairs of numbers: the eight edges in the drawing on the left are 1-2 (which is the same as 2-1), 1-4, 1-6, 3-2, 3-6, 5-2, 5-4 and 5-6. I'm sure you'll see that they are the same as the edges in the other drawings.

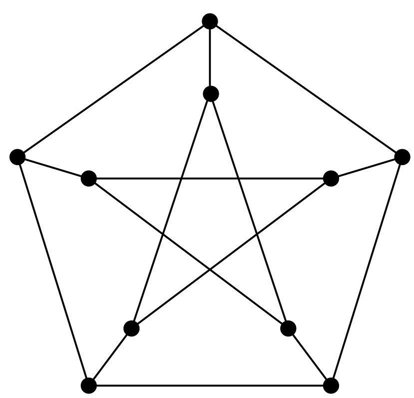
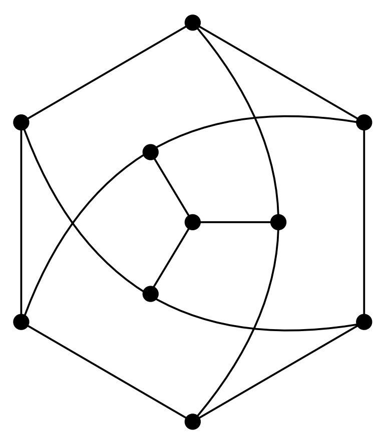






Wait! Did I say geometry plays no role? It does, to some extent. The two drawings on the left have crossing edges and the third does not. A graph that can be drawn on a flat surface with no crossing edges is called a planar graph. The graph above, in any of its forms, is therefore planar. A graph that cannot be drawn without crossing edges is called nonplanar. (Interestingly, some nonplanar graphs can be drawn on other surfaces, like the surface of a torus or doughnut, without crossing edges.)

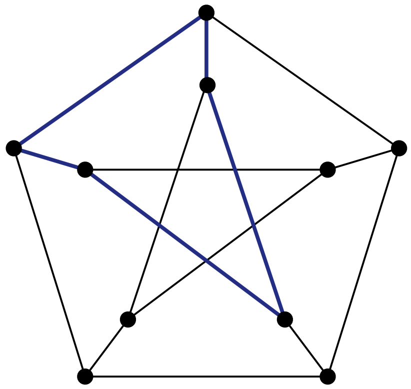
Some graphs are important enough to have names: the famous Petersen graph is sketched below. You may wonder which of the two is the Petersen graph. Both are; they are just different drawings of the same graph. Try to prove this by labelling the 10 vertices of each graph with the labels 1, 2; ..., 10, and check that you get the same edges, as explained above. The drawing on the left is the most common drawing of the Petersen graph; there are several others.

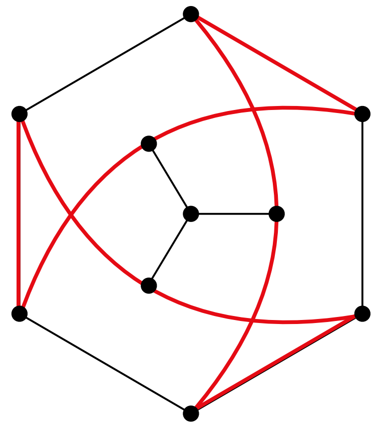



The Petersen graph has many interesting properties. It is cubic: each vertex is the meeting point of three edges. Cubic graphs are quite common, but only occur when the graph has an even number of vertices. (Why?) Try to colour the edges so that the three edges meeting at any vertex all have different colours. Even your third cousin twice removed will tell you that you need *at least* three colours, but can you do it with three? I can't! I can do it with four colours, though. In fact, Vadim Vizing, a Russian mathematician, proved in 1964 that the edges of any cubic graph can be coloured as described using either three or four colours. (He actually proved much more than this.) Because we need four colours to colour the edges of the Petersen graph, we say it has **chromatic index 4**.

A cycle in a graph is obtained as follows: begin at a vertex and follow an edge to a new vertex. Follow another edge to yet another new vertex, and continue (using only new vertices) until you reach the very first vertex again. If a cycle has k vertices and therefore also k edges, where $k \geq 3$, it is called a k-cycle. A 5-cycle and a

9-cycle of the Petersen graph are pictured below. A cycle that contains all the vertices of the graph is called a **Hamilton cycle** (after the Irish mathematician Sir William Rowan Hamilton, 1805-1865). The Petersen graph doesn't have a Hamilton cycle. Try to prove this by thinking about how many times such a cycle would have to cross the edges joining the exterior pentagonal 5-cycle to the interior star-shaped 5-cycle in the sketch on the left, and examining a few cases. The Petersen graph also doesn't have a 3-cycle, nor a 4-cycle.





Stare at the Petersen graph a little longer. How many edges do you need to remove to split the graph into two parts that can be pulled apart, with no edges between the two parts? Such parts are called the components of the resulting graph. Well, obviously it can be done by deleting three edges incident with the same vertex. Let's then phrase the question a little diferently: how many edges do you need to remove to split the graph into two components, each of which contains a cycle? If you have your wits about you, you will say right away that, since the Petersen graph doesn't have 3- or 4-cycles, it will have to be split into two 5-cycles, and in order to do this, you have to remove five edges. If k or more edges need to be removed to split a graph into two components, each of which has a cycle, we say the graph is cyclically k-edge connected. The Petersen graph, therefore, is **cyclically 5-edge connected.**

Another thing you can try to do is to draw the Petersen graph without crossing edges. A good approach is to label the vertices of the 9-cycle in the figure with the numbers 1, 2,...,9. Naturally, this cycle can be drawn with no edges crossing - simply draw it as a nonagon. Add the tenth vertex (inside or outside the nonagon, it makes no diference) and join it to the appropriate vertices. Now try to add the rest of the edges... it won't work! The Petersen graph is **nonplanar**.

As you may have guessed by now, the Petersen graph is an example of a snark: a snark is, by definition, a

- cyclically 4-edge connected
- cubic graph
- with chromatic index 4.

They were named snarks in 1975 by Martin Gardner, an American popular mathematics writer, because, like Lewis Carroll's snark, they were so hard to find. Snarks have no Hamilton cycles and are nonplanar. The former is easy to see: if a cubic graph has a Hamilton cycle, we can colour the edges of the cycle alternately with two colours (because it has an even number of edges), and then colour all the remaining edges with a third colour. The fact that snarks are nonplanar is a deep, difficult-to-prove result. It is a consequence of the Four Colour Theorem, and now we're back to the origin of snarks.

The **Four Colour Theorem**, in its modern form, states that the vertices of any planar graph can be coloured, using four or fewer colours, so that adjacent vertices (i.e., vertices joined by an edge) have diferent colours. The Four Colour Conjecture (4CC) was first posted by Francis Guthrie (1831-1899) in 1852, and, to make a long story short, eventually proved in 1976 by Appel and Haken, with much help from a computer and a program written by Appel's doctoral student Koch. In 1880, the Scottish mathematical physicist Peter Tait proved that the Four Colour Theorem was equivalent to the statement that no snarks were planar. But - in 1880, there were no known snarks! The Petersen graph was the first snark to be discovered. It was named after the Danish mathematician Julius Petersen, who presented it in the 1890's as counterexample to Tait's claim that all cubic graphs were 3-edge colourable. However, Alfred Bray Kempe, who also tried to prove the 4CC, already mentioned this graph in 1886.

Snarks number 2 and 3 were discovered by the Yugoslavian mathematician Danilo Blanuša in 1946, 60 years later. Each has 18 vertices (and how many edges?) and can be obtained from two copies of the Petersen graph by deleting two edges not sharing any vertices from one copy, two adjacent vertices from the other copy, and joining the two parts in two different ways.

Only two years later, the fourth snark was discovered by Blanche Descartes in 1948. It has 210 vertices and can be obtained from the Petersen graph by replacing each vertex with a nonagon and each edge with a graph obtained from the Petersen graph by deleting two nonadjacent vertices. (Blanche Descartes was the collective pseudonym of R. Leonard Brooks, Arthur Harold Stone, Cedric Smith and William Tutte.)

The years passed (as they are wont to do). Snark number 5 with 50 vertices was discovered in 1973 by the Hungarian-Australian mathematician George Szekeres. It is formed by deleting two edges from each of five copies of the Petersen graph and joining the resulting graphs in a specific way. Finally, in 1975, the American game theorist Rufus Isaacs discovered the first infinite class of snarks, called the flower snarks. Even they have features similar to the Petersen graph, and so have all other snarks discovered since.

William Tutte conjectured in 1966 that every snark contains the Petersen graph in some well-defined way; in mathematical terms, every snark has a **Petersen minor**. A proof of this conjecture was "announced" by Robertson, Sanders, Seymour, and Thomas in 1999, but they said it was too long to write up properly for scrutinizing by the mathematical community.

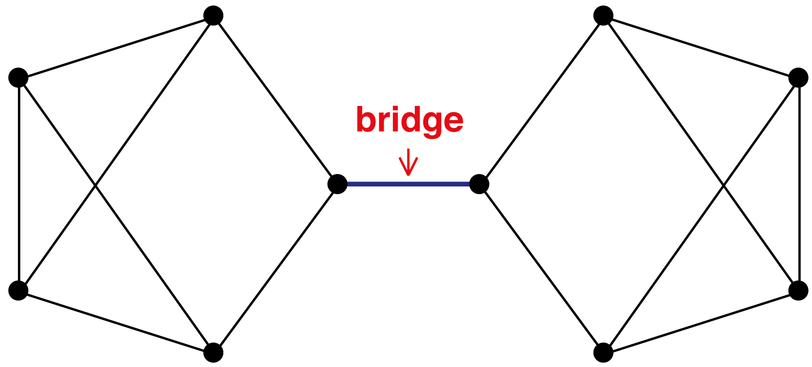
By 1981, there were eight known snarks having 30 or fewer vertices. Then computers entered the picture. Eventually, in 2013, Brinkmann, Goedgebeur, Hägglund & Markström generated all snarks up to 36 vertices. Amazingly, there are more than 64 million of them, as tabled below.

ORDER	10	18	20	22	24	26	28	30	32	34	36
# SNARKS	1	2	6	20	38	82	2,900	28,399	293,059	3,833,587	60,167,732

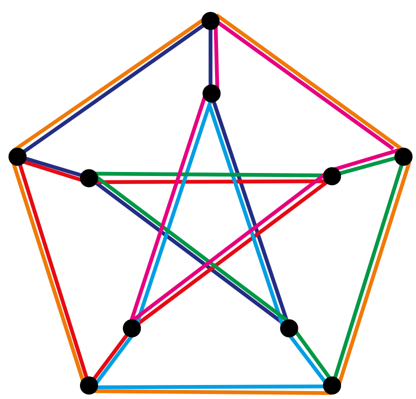
If the interest in snarks was sparked by the 4CC, which was solved over 40 years ago, why are people still looking for them? Of course, they are mathematically interesting, and mathematicians are interested in ... well, mathematically interesting objects! But snarks are more than just interesting; they are important for other reasons.

One of the most important unsolved problems in graph theory is the **Cycle Double Cover Conjecture**, which states that **every bridgeless graph has a cycle double cover**:

- An edge of a graph is called a bridge if its removal results in a graph having more components than the original, as illustrated below.



- A cycle double cover of a graph is a collection of cycles (in which the same cycle may appear twice) such that each edge of the graph belongs to exactly two cycles in the collection. A cycle double cover of the Petersen graph is shown below.



There are a number of other related conjectures too. *And each of these conjectures will be true for all bridgeless graphs if they prove to be true for snarks!* Brinkmann et al. checked the Cycle Double Cover Conjecture and other related conjectures for all the snarks they generated, and all are true for those specific snarks. But no matter how many snarks people generate and check, there remain infinitely many of them that cannot be checked. Only a theoretical proof will work.

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The Beauty of Quadratic Equations

BY SUHRID SAHA

Suhrid wrote this article when he was a high school student in Greenwood High, Bangalore, India. He is currently studying Mathematics and Computer Science at University of California at Berkeley . He has a deep interest in pure math and plans to become a professor someday. In his free time, he is an avid cinephile who loves sitcoms and superhero movies.

Figure 1

Figure 1 shows the first page of Al-Khwarizmi's popular treatise on Algebra written in early 9th century AD[i] .

Al-Khwarizmi's book (*Al-Jabr*) became famous for providing a systematic method to deal with linear and quadratic equations. It starts by providing a geometrical method to solve quadratic equations. This was partly known to some Indian mathematicians like Brahmagupta. Let me show you how he did it by solving a quadratic equation from the book.

Figure 2 shows a quadratic equation problem Al-Khwarizmi solves in his book, using the completion of squares or "balancing" method (you'll see why it is called this, soon). Although the language used is entirely rhetorical (you won't find any symbols or numbers), I'll write the problem mathematically and show you how he solved it.

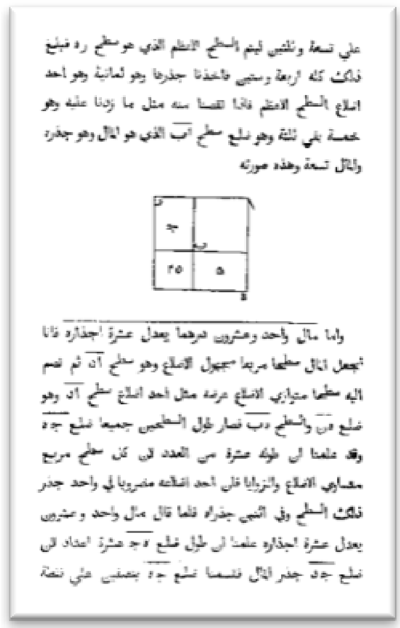

Figure 2

Consider: $x^2 + 10x = 39$

To solve this, we start by drawing a rectangle with sides x and $x + 10$. Evidently the area of the rectangle is the product of the length (x) and breadth ($x + 10$), so the area will be the left-hand side of the quadratic equation ($x^2 + 10x$). We then divide the rectangle into a square of length x and rectangle with dimensions 10 and x.

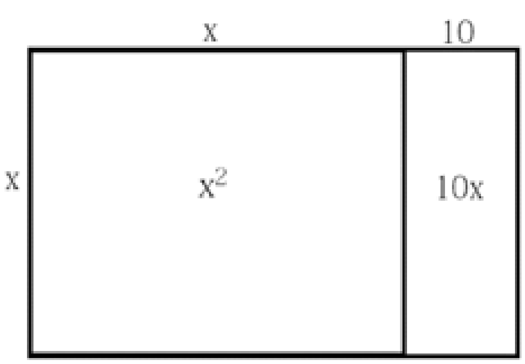

Figure 3

We then cut the 10 x rectangle in half and move one half below the square of length x (Figure 4a). The area of this figure has to be 39, since from the problem we are given that $x^2 + 10x = 39$.

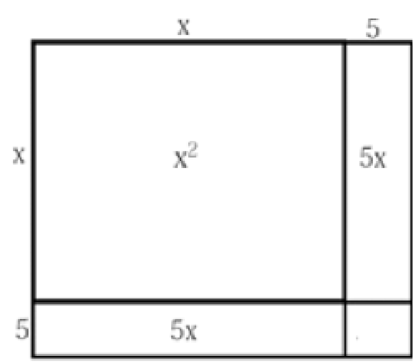

Figure 4a

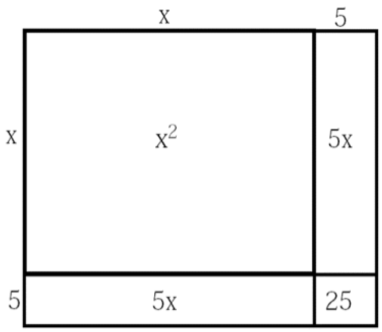

Figure 4b

However, now we are left with an incomplete portion of a square, two of its sides being x+5. The easiest way to complete it is to add a square of length 5 (shown in Figure 4b).

The area of the figure without the new square (with sides of dotted lines) is 39, so along with the new square it must be $39 + 5^2 = 39 + 25 = 64$. So, we now have the area of the full square to be 64 and side length to be $x + 5$, so we can write $(x + 5)^2 = 64$

$$\Rightarrow x + 5 = \pm\sqrt{64} = \pm 8 \Rightarrow x = 3 \text{ or } x = -13$$ [1]

The idea above was to take the constant term to the right-hand side ($x^2 + 10x = 39$ is the same as writing $x^2 + 10x - 39 = 0$, so here we take 39 to the right) and add a constant to both sides of the equation (25 in this case, hence the square with side length 5) such that the left hand side becomes the square of a linear polynomial ($x^2 + 10x + 25 = (x + 5)^2$ in this case). We can use this method to solve any quadratic- you can try using it to solve $x^2 + 12x = 45$, $4x^2 + 4x = 3$. Now we will move towards the general quadratic (finally!).

Consider the general form of a quadratic equation: $ax^2 + bx + c = 0$ where $a \neq 0$ & a, b, c are any real numbers. It is easily solvable using the completion of squares method as shown above.

$$ax^2 + bx + c = 0$$

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \text{ (Dividing by a)}$$

$$\Rightarrow x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Rightarrow x^2 + 2 \cdot \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \text{ (The most important step!)}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \text{ (Remember the } \pm \text{ sign!)}$$

[1] Al Khwarizmi however did not consider the negative solution of -13. If you think about it from the perspective of geometry, the negative solution is of no physical significance since the length of a side can't be negative, but we know a quadratic must have two roots and hence it makes sense only if we write both solutions to cover all cases.

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ (QUADRATIC FORMULA!)}$$

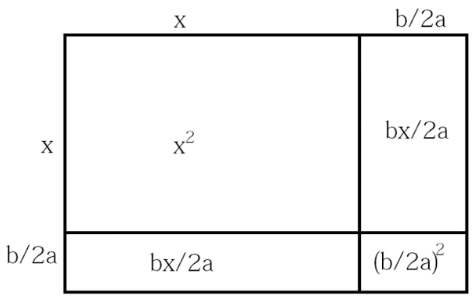

Figure 5 (for quadratic equation above)

And thus, we have derived the formula for solving the general quadratic and provided a standard method for solving all quadratic equations.

This technique is taught in high-schools around the world as the method of “completion of the square”.

If you really enjoyed these derivations, you should definitely learn ways of solving equations of higher degrees i.e. cubic (3rd degree) and quartic (4th degree) equations.

The most commonly known method to solve cubic equations is Cardan’s method . There are also other methods to solve a cubic equation- Lagrange’s resolvent, Viète’s trigonometric solution, Bombelli’s solution, etc. Each of these involves amazing ideas, have interesting solutions, and yet fail in certain cases.

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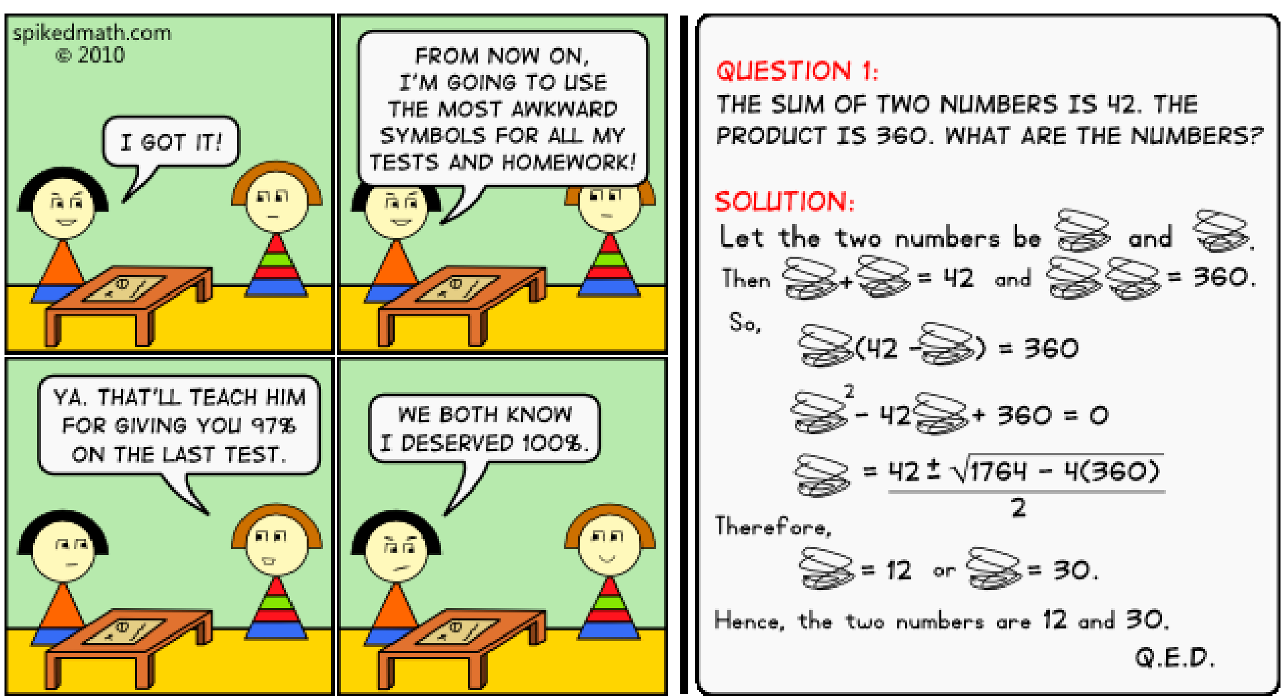


The tale of two coins

BY KAROL GRYSZKA

Karol Gryszka is an assistant professor at the Pedagogical University of Krakow in Poland. His research is the study of asymptotic behaviour in dynamical systems. He is particularly interested in popularizing mathematics with articles, lectures, workshops and school projects.

THE CASE OF COINS

Alice and Bob want to play a coin game. The main requirement is that the coin has to be fair, that is there is exactly 50% chance to obtain either side. To test this they decide to roll the dice 250 times. Their results are presented below:

heads 134
tails 116

Alice sees these numbers and begins to question the fairness of the item. She thinks that if each side has the same probability, then the numbers should differ less. Bob, on the other hand, thinks that the heads side might have just been lucky and there is nothing unusual with the numbers. We could ask who was right, but that is difficult to answer, especially if we have no prior knowledge about the coin. Let us look more carefully at this problem.

Take any coin from your wallet and investigate it. There are some markings on both sides, each one is different and has a unique look. Then there are dents at the edge and sometimes they are placed irregularly. That combination makes the coin not perfectly (mathematically) fair - it is not perfectly symmetric, the weight is not equally distributed and so on. However, we tend to call these coins fair and we use them to randomize a simple situation with 50 : 50 chances. We do this because we are used to selecting one of two outcomes using a coin flip, and we also think this is a convenient and fair way to do this.

John von Neumann (see [JvN]) created a simple algorithm that, in practice, allows us to ignore any possible bias (that is the deviation from equal chances for either side) of the coin. The procedure is very simple. Let us flip the coin twice and record T for tails and H for heads. There are 4 possible results, arranged in pairs: (T,T), (T,H), (H,T), (H,H). We read the result using the following table:

$$\begin{array}{ll} \mathrm{HH} & \to \text{repeat} \\ \mathrm{TT} & \to \text{repeat} \\ \mathrm{HT} & \to \text{heads} \\ \mathrm{TH} & \to \text{tails} \end{array}$$

In other words: if the letters are different, we read the first one and call it the final result. If the letters coincide, we repeat the procedure

Let us analyse why this procedure is correct. Assume that the probability of recording H is p, then there is exactly $p \cdot (1 - p)$ chance to see the pair (H, T) and $(1 - p) \cdot p$ to see (T, H). So the chances are equal and therefore the algorithm produces H and T with equal probabilities. The repetitions make sure that we eventually finish with heads or tails. Even though one may think it is possible to "get stuck", that is to continuously flip either only heads or only tails, this is impossible. The chance of always flipping the same side infinitely many times, is equal to zero. With this in mind, we can conclude that there are only two possible outcomes: heads or tails and they share the same chance, so the chance is exactly 50%. Thus the algorithm works as intended. And that's it - You no longer have to delve into imperfect coins. All you need is to flip the coin twice and see what the first result was. The kind of coin does not matter either - Alice and Bob can switch coins between each pair of flips and they still have a perfect 50% chance to flip heads or tails.

IS IT HARDER FOR DICE?

Alice and Bob want to know if the idea presented above can be extended to more items that randomly assign one outcome. They now take a die and look at it. We also look. The die may not be perfectly balanced (the distribution of material may not be uniform) and there are dots made of different material with different numbers on each face which make the dice even more non-symmetrical. On the other hand, the cube is a perfect shape to randomize one number from the set {1, 2, 3, 4, 5, 6} (or any other set having 6 elements) -

it is symmetric, all vertices, edges and faces are the same. It is the way dice are manufactured that makes them imperfect.

Bob thinks that what he has just learnt about the coin can be carried to dice. Here is how he does it. First, we need to know what a permutation is. A **permutation** is any arrangement of the given items (mugs on the table, students in the classroom, or abstract objects as well, such as numbers or letters).

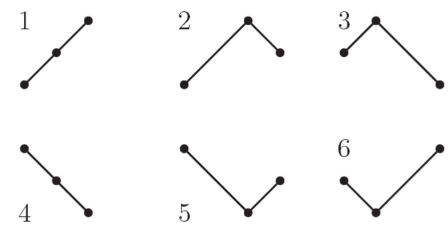
In any permutation, each item occurs once. Two numbers 1 and 2 can be arranged in two permutations denoted by 12 and 21. Three numbers 1, 2 and 3 have 6 permutations: 123; 132; 213; 231;312 and 321. On the other hand, the arrangement 133 is not a permutation - 3 appears twice, which is forbidden by the rules.

Bob should roll the dice **six** times and hope for different numbers - one of the 720 permutations of the set of numbers from 1 to 6. If he sees different numbers, he calls the final result the first rolled number, otherwise he repeats the algorithm. For example, if he sees 354162 (each number appears once), then the number 3 is chosen. But if he sees 456314, he rejects the sequence (4 appears more than once and 2 does not appear at all), and repeats the algorithm.

Why does this idea work? Alice says only the permutations matter, since the rest is discarded. Then she notices that the order of numbers does not matter - the chance to obtain each specific permutation of six elements of biased dice is always the same. Why? Lets say chances for each side 1 to 6 are p_1, p_2, p_3, p_4, p_5 and p_6 respectively. If we see 354162, then the chance for that is $p_3\,p_5\,p_4\,p_1\,p_6\,p_2$. If we see 123456, the chance is $p_1\,p_2\,p_3\,p_4\,p_5\ p_6$ - the same number with different order of components. Since the same rule applies to any permutation, all permutations have the same chance. Since there are the same number of permutations starting with each number, the algorithm works as intended.

Patience is a key word here. Even if the dice is a fair one (each side has 1/6 chance), then on average only 1 of 65 repetitions with 6 rolls each will result in a permutation. Things can get much worse if the deviation from 1/6 is greater, but a priori one cannot expect to know what is the exact chance for each side and therefore what is the exact number of dice rolls.

The extension of von Neumann's algorithm suggested by Bob has a major flaw as described above - lots of repetitions needed to obtain a permutation. Alice thinks about this and suggests the following idea (see [CPC]): she rolls the dice three times and tracks how the numbers change with the help of the following diagrams.



She explains it using these examples: if you receive (1, 4, 5), then the second number is greater than the first and the third is greater than the first and the second. This is illustrated by diagram labelled by 1, so we assign 1 as the final result. If you receive (1, 5, 4), then the second number is greater than the first and the third is between the two. This is illustrated by diagram 2, so we assign 2 as the final result. If you receive (4, 5, 1), then the second number is greater than the first and the third is smaller than both the first and the second number. This is illustrated by diagram 3, so we assign 3 as the final result. The same principle applies to the remaining diagrams which correspond to: (5, 4, 1) (diagram 4), (5, 1, 4) (diagram 5) and (4, 1, 5) (diagram 6).

To sum up - we check the relative ordering of all numbers and see which diagram it matches. Should the same number occur twice, all three rolls should be discarded, and the process should be started over until three distinct numbers are rolled.

The idea presented by Alice works due to a clever division of all 3-dice roll arrangements into 6 patterns and the patterns which are discarded. If the dice aren't fair, the different ways to achieve each pattern may have different probabilities (for example 123 may be more likely than 234), but any choice of three different numbers can be permuted to match one of 6 diagrams. For example if the numbers rolled are 1, 2 and 3 in some order, the table below describes all possibilities and their corresponding diagrams.

$$\begin{aligned} 123 &\to 1 \\ 132 &\to 2 \\ 231 &\to 3 \\ 321 &\to 4 \\ 312 &\to 5 \\ 213 &\to 6 \end{aligned}$$

So from the numbers 1, 2 and 3, each of the six patterns is equally likely to occur; and it is the same with any other three numbers. This means that each pattern is equally likely to occur.

The rejection rate for Bob's algorithm is quite high- we have already mentioned that in 6 rolls, there is roughly a 1 in 65 chance of obtaining a permutation, which gives $\frac{319}{325} \approx 98.46\%$ rejection rate for a perfectly fair dice. This is awful. Alice's algorithm is much better - the rejection rate is $\frac{4}{9} \approx 44.44\%$ Not only does it have a 30 times higher acceptance rate, but it also requires fewer rolls per attempt. An outstanding improvement, indeed. You can make even better assignments, but we won't go into details.

EXTENDING TO ANY NUMBER OF EQUAL OUTCOMES

Alice now looks at the coin again and she starts thinking about the following problem: can we use a coin to obtain one number from the set {1, 2, 3 }? This shouldn't be hard, she says.

Is Alice right? Let's think for a moment - we have already seen coin flips that provided three outcomes: heads, tails and repeat. However these were imperfect for Alice's problem: repeat can have a different chance than the other two have. At the same time, these outcomes were assigned to a different problem - removing the bias from the coin.

We shall assume that the coin we are about to flip is fair. It is a valid assumption to make as we have already solved that issue earlier. If the coin is fair, we can again flip it twice and read the result according to the table:

$$\begin{aligned} \mathrm{HH} &\to 1 \\ \mathrm{HT} &\to 2 \\ \mathrm{TH} &\to 3 \\ \mathrm{TT} &\to \text{repeat} \end{aligned}$$

Alice was right! All we needed was a proper approach and a simple correction of what we have already learned.

What if Bob asked the same problem but for the set {1, 2, 3, 4, 5}? Easy! Look at the table:

$$\begin{aligned} \mathrm{HHH} &\to 1 & \mathrm{THH} &\to 5 \\ \mathrm{HHT} &\to 2 & \mathrm{THT} &\to \text{repeat} \\ \mathrm{HTH} &\to 3 & \mathrm{TTH} &\to \text{repeat} \\ \mathrm{HTT} &\to 4 & \mathrm{TTT} &\to \text{repeat} \end{aligned}$$

What if Bob asked for the set {1, 2,...,22}? Things get a little more complicated, but we can still do it. We use the same principle again and toss the coin 5 times:

$$\begin{aligned} \mathrm{HHHHH} &\to 1 & \mathrm{THTHT} &\to 22 \\ \mathrm{HHHHT} &\to 2 & \mathrm{THTTH} &\to \text{repeat} \\ \dots &\to \dots & \dots &\to \dots \\ \mathrm{THTHH} &\to 21 & \mathrm{TTTTT} &\to \text{repeat} \end{aligned}$$

How were these assignments created? Look at the 5-digit numbers composed of zeros and ones and write them down in increasing order: 00000, 00001, 00010, . . ., 11101, 11110 and 11111. Then just replace 0 with *H* and 1 with *T* and assign with each sequence of letters a number, starting from 1 until 22. The remaining sequences receive repeat values.

And that's it. Alice and Bob can select any number from any set provided each number has equal chances. And it doesn't matter if the coin is fair or biased.

THE CLOSURE

Alice and Bob can take the coin game even further. There is one problem related to the one above, which includes irrational probabilities (for instance $\frac{\pi}{4} \cdot 100\% \approx 78.54\%$ chance for one number). It turns out there exists an algorithm based on the binary representation of a number, which allows Alice to select any number of outcomes with any probabilities. You can read more about this in [AI].

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A Number Guessing Game

BY SAYAK CHAKRABARTY

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Abstract

We will start with an old number guessing game between two players. We discuss the salient features of this game and what happens when we slightly alter the rules of the game. Finally, we give the strategy to play the game optimally.

1 INTRODUCING THE GAME

Suppose we have two players, A and B, and we have told them two consecutive positive integers. While A and B know their numbers are consecutive, each of them only knows his/her own number. For instance, if A has the number 4, she knows that B has either 3 or 5. The game is that they have to guess the other number by repeatedly asking the same question of their opponent, and the question is "Do you know my number?". This is the only question that they can ask each other and the answer should be either "No" or "Yes" (and they answer truthfully). The game is to be the first to learn the other persons number.

THE STRATEGY

Since the numbers are consecutive, if A is given the number k then the only thing to worry about is whether B's number is k +1 or k - 1. Now let us go through the following dialogue and find out what they are actually thinking, supposing that they speak aloud.

A: "Do you know my number?"
B: "No. Do you?"
A: "No. If B had 1, then obviously he could have guessed my number as 2, since they are positive integers. So, B has his number greater than or equal to 2. Do you know my number?"
B: "No. A knows my number is greater than or equal to 2. If A had 2, then he would have guessed my number as 3 correctly. So, A's number is greater than or equal to 3. Now do you know my number?"
A: "No. But B knows my number is greater than or equal to 3. So, if B's number had been 3, he would have correctly guessed my number as 4. So, B's number must be greater than or equal to 4. Now do you know my number?"

They continue in this way, at every stage concluding their opponent's number 1 bigger than what they already knew. Now suppose A had his number as k and B had $k + 1$. Then, in the course of the game, as soon as A concludes B's number is greater than or equal to k, the game is over since he now answers B's question positively and says B's number is $k + 1$. And interestingly, B also concludes from this answer that A's number was k. So, the game has a nice strategy!

2 PROPERTIES OF THE GAME RULE

Before we start generalizing, we note a few very important observations which made the strategy work, and this is the key to all generalizations.

- For the strategy to start working we need a lower bound, which was 1 in our case, since we are dealing with positive integers. Without this, A could not start thinking what B might have or not.
- Both A and B know the rule that the numbers are consecutive. This is extremely important because if someone's number is 1, they know the other number. So given the lowest number, the other number is easy to determine.
- The person who has the smaller number, first gets to know the other number. After they answer the opponent's question truthfully, the other one gets to know the missing number too. (So if the person with lower number lies, then they will always win and their opponent will keep searching, but we do not consider situations like that in this article.)
- Even if A and B do not think aloud and do those calculations mentally, after every time B says "no" to A's question, A knows exactly what calculation B has made and same for the other way.
- The last and most interesting question is: Suppose a third person is present while A and B

are playing the game. This person knows that the numbers are consecutive, and suppose A and B do not think aloud i.e. they only answer "yes" or "no" unlike the dialogue we saw above. Then, as soon as A and B know their numbers, does the third person also know their numbers?

The answer to the question is. If we consider both A and B each giving an answer as a round, then after the first round the third person knows that both of them have number greater than or equal to 2. He keeps track of the number of rounds and as soon as someone answers positively, he knows the number and also knows the one who answers first has the lower number. But now, we change this rule a bit to make things hard for the third person.

3 NUMBER GUESSING WITH A DIFFERENT RULE

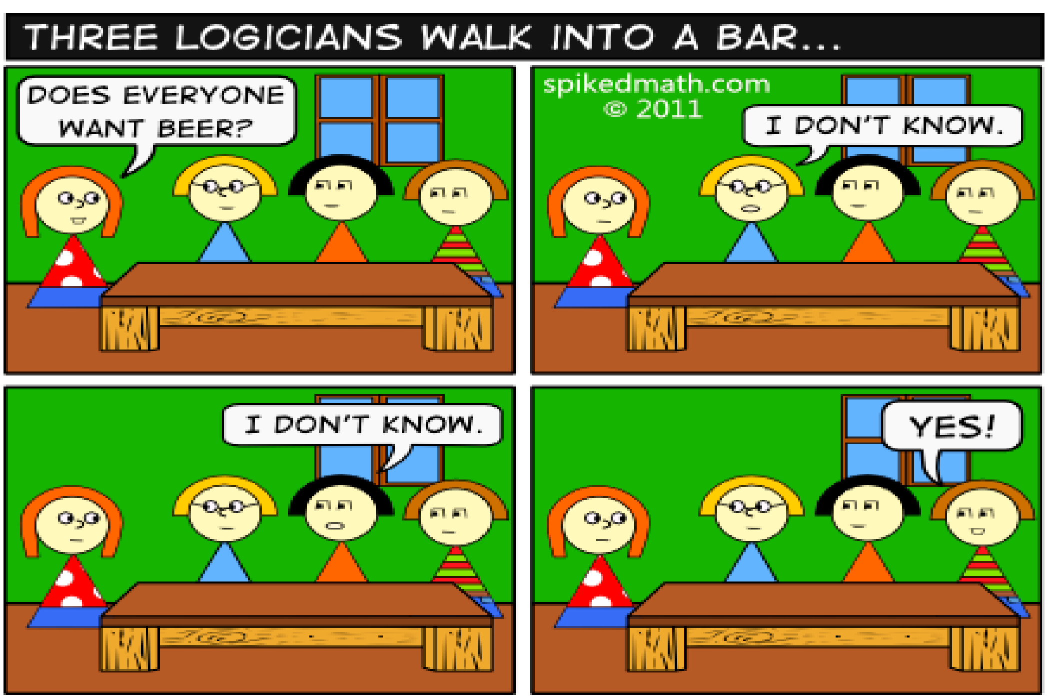
Suppose we have two players, A and B, and we have told them two integers which are sufficiently large, and that one is double of the other. The game is that they have to guess the other number, by asking repeatedly the same question to the opponent, and the question is, "Do you know my number?". This is the only question that they can ask each other, and the answer should be either "No" or "Yes" (and they answer truthfully). Then they ask in return again, "Do you know my number?". The puzzle is whether they will be the first to learn the other persons number.

THE STRATEGY

The interesting thing to note here is that the strategy is nothing new and the actual underlying problem is the same. For example, lets say the two numbers are A = 20000002 and B = 40000004.
A says to B: "Do you know my number?" (at this point B thinks A=2000002 or 8000008).
B says to A: "No. Do you know my number?" (A knows that B has 1000001 or 4000004; when A hears the "no", she realizes that B does not have 1000001, so that B must have 4000004). The answer, then, is "Yes - it's 4000004".

4 CONCLUSION

As an exercise, the interested reader can try to extend this game to 3 or more players with the same rules. The same strategy works with very little variation in reasoning, though we have to take into account the extra players in this case. This game also has interesting applications in secret sharing, a method used in cryptography. Here again, the interested reader can find out if the last property (involving the third person who is eavesdropping) of section 2, still holds when the game involves more than 2 players, and also when the rule is changed slightly as in section 3.



The Ancient Indian Poornam Mantra and the Paradox of Infinity

BY KAPIL KRISHNA

Kapil Krishna is a high school senior in Virginia, United States. He loves mathematics and wishes to pursue a career in engineering. Outside of school, he likes to spend time playing football and tennis.

INTRODUCTION

Infinity is a concept which has mesmerized mathematicians for centuries. A concept which seems straightforward when first taught, but becomes progressively more complex through a deeper understanding. Yet more than four thousand years ago, the concept of infinity had been succinctly explained in Indian Vedic literature. The word Vedas translates to "knowledge" and is a collection of hymns, poetry, and Hindu ceremonial formulae. Interestingly, the knowledge of mathematics was deemed to be so important that it was formulated into mantras recited daily. [1] Here we look specifically into mentions of infinity in the Isha Upanishad, and how it connects to the overall paradox of infinity discovered in modern times.

POORNAM AND INFINITY

The Upanishads are parts of the Vedas that contain central concepts and ideas of Hinduism [2]. The Isha Upanishad is one of the shortest Upanishads, embedded as the last chapter of the Shukla Yajurveda. [3] It starts with the following famous verse:

Om
Purnamadah Purnamidam
Purnat Purnamudachyate
Purnasya Purnamadaya
Purnameva Vashishyate
Om Shanti, Shanti, Shanti

The line by line translation of this is as follows:

Om.
That is infinite, this is infinite;
From That infinite, this infinite comes.
From That infinite, this infinite removed or added;
Infinite remains infinite.
Om. Peace! Peace! Peace! [4]

The lines from this verse show something interesting. The conception that when infinity is taken from infinity, the infinite is remaining. This is perplexing because it contradicts common arithmetic. This is because we have always been taught that when a quantity x, is subtracted from itself, the result always results in zero. However, the Upanishad says otherwise when it comes to the concept of infinity. Rather, when the infinite is taken away from the infinite, the infinite is not reduced in any way. That is simply because taking away from the infinite is not possible. In Indian Vedas, this concept applied more to the creation and the absolute, to show that when the infinite of the origin is taken for a new creation, the full is never affected. [5] However, it is quite remarkable to see how this conceptualization of infinity is still applicable to modern mathematics.

THE INFINITE HOTEL PARADOX

A German mathematician by the name of David Hilbert (1862-1943) proposed a thought experiment known as the Infinite Hotel. The experiment shows the problems that occur when infinity is used as a number rather than a concept. The experiment starts with a hotel with an infinite number of rooms which are all full. These rooms have each been numbered with the set of natural numbers (1,2,3...) However, when a new guest arrives each existing guest simply moves down a room, making space for the new guest. When twenty guests arrive, each previous person moves down twenty room positions. This can also be done if an infinite busload of new guests arrives: each previous guest goes to the room number double theirs (i.e. the person in room number two moves to room number four) This allows half the rooms to be emptied and the infinite bus of people are all accommodated [6]. This experiment showed that it was possible for an operation to be performed on infinity, by accommodating more guests and still resulting in infinity. This is essentially the same conclusion that is shown in the Isha Upanishad stating that even when infinite is removed or added, the result is infinite.

INFINITY THROUGH CARDINALITY

By understanding how cardinality (roughly the number of elements in a set) applies to infinite sets, we will see how a set of natural numbers can have the same cardinality as two distinct copies of the natural numbers. Two sets are said to be of the same *cardinality*, if there is a one-to-one correspondence between them. For example, the even natural numbers have the same cardinality as the set of natural numbers, because we can pair each even number $2n$ with the natural number n. (This gives a way of matching up the even natural numbers with the natural numbers where each even number is paired with one natural number; and each natural number is paired with one even number). Similarly, the odd natural numbers have the same cardinality as the natural numbers: we can pair the odd number $2n$-1 with the natural number n. But now, the natural numbers are made up of two subsets, each of the same "size" as the natural numbers themselves.

The parallel with the verses in the Isha Upanishad is inescapable.

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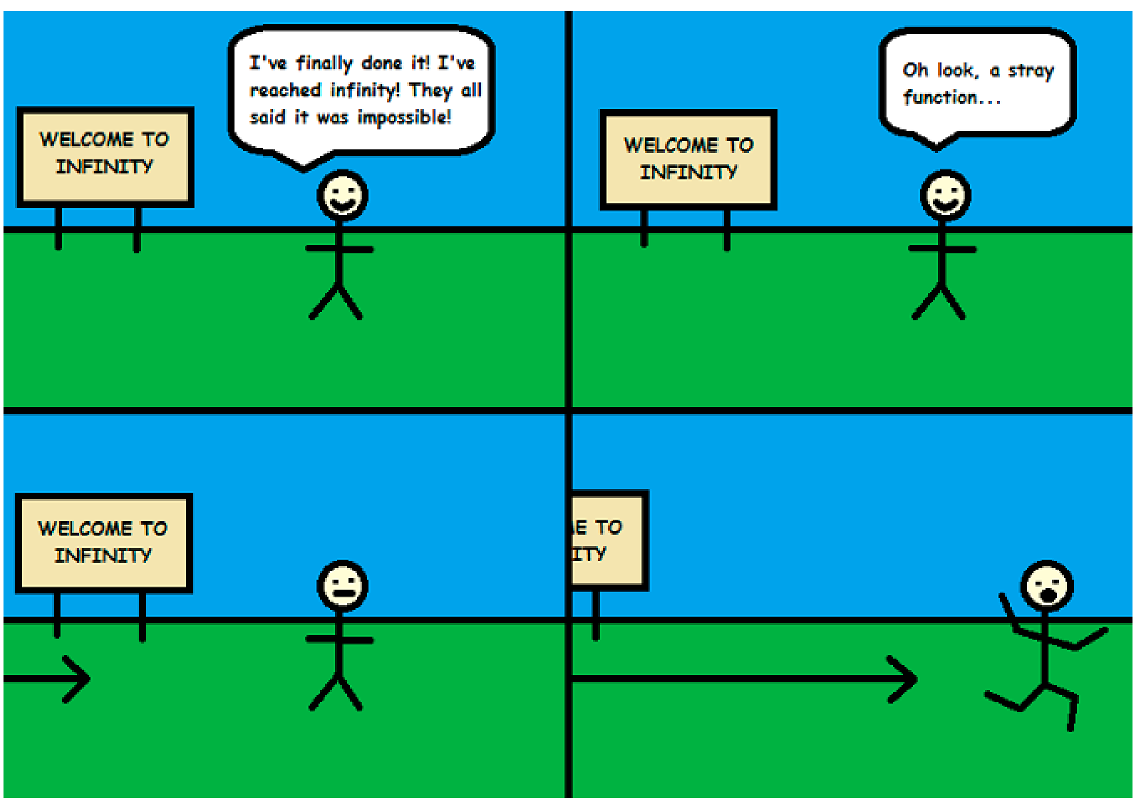


PI22 MATH QUICKIES

Solutions will be published in the next issue of PI IN THE SKY

1 If $\frac{1}{13} = 0.a_1 a_2 \ldots a_n \ldots$, find the sum of $a_1 + a_2 + \cdots + a_{2021}$

2 On an 8 x 8 checkerboard, how many rectangles (and squares) are there, that are made up of a number of whole squares of the board?

3 Find the numbers of integers x between 10 and 99 (inclusive) which have the property that the reminder of x^3 divided by 100 is equal to the cube of the unit digit of x.

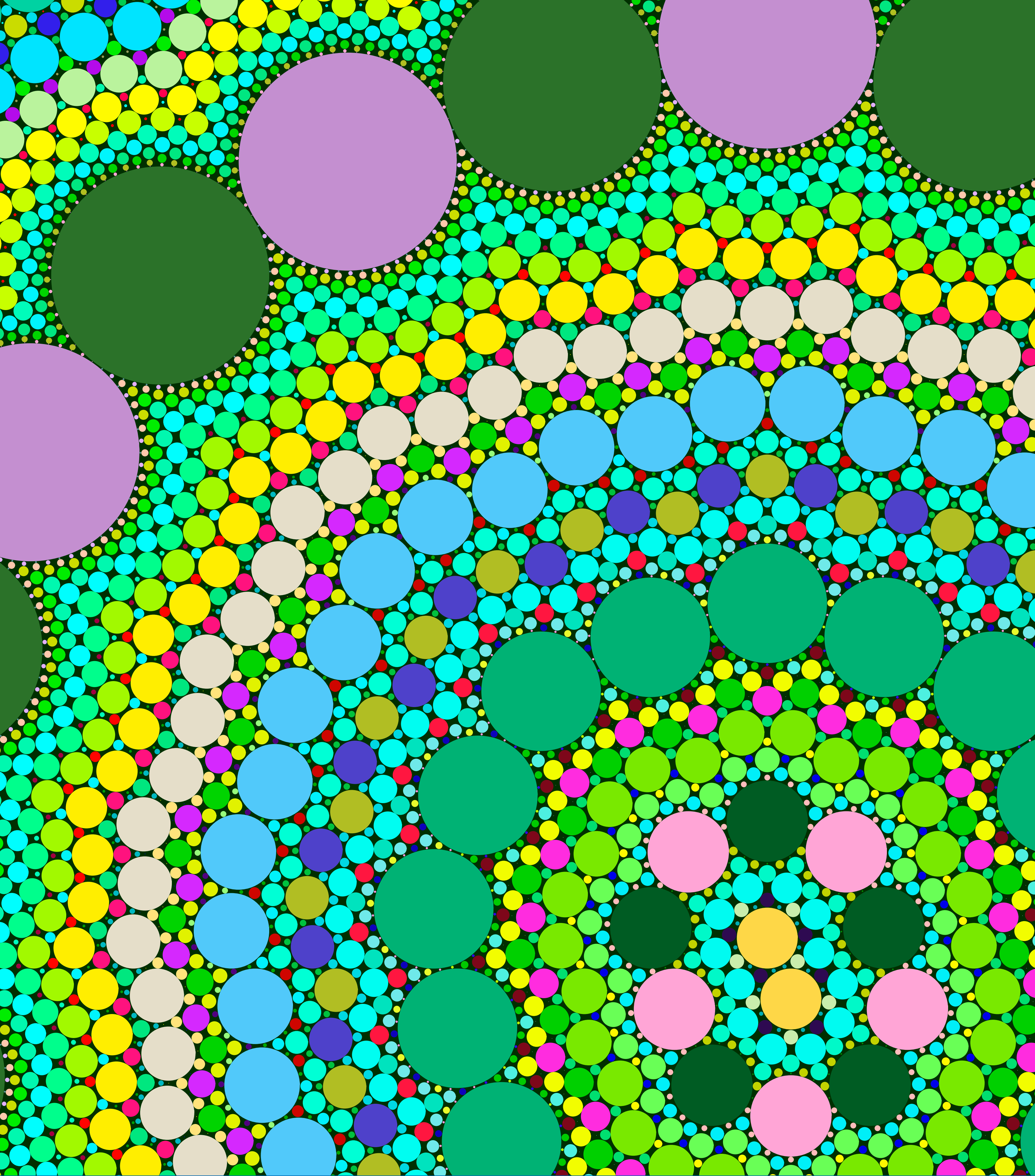
4 A number M has the property that if x and y are any positive numbers such that $2x + 3y \leq M$ then $x \cdot y \leq M$. Find the maximum possible value of M.

5 Is there any integer n such that $(\sqrt{3} + \sqrt{2})^{2020} = \sqrt{n} + \sqrt{n-1}$?

6 Find all pairs of integers (x, y) such that $x^2 + y^2 = 35(x + y)$.

PRIZE!

PIMS is sponsoring a prize of $100 to the first high school student (from within the PIMS operating region: Alberta; British Columbia; Manitoba; Saskatchewan; Oregon; Washington) who submits the largest number of correct answers before March 1, 2021. Submit your answers to: pims@uvic.ca



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